Thurston's Geometrisation "Conjecture"

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Outline

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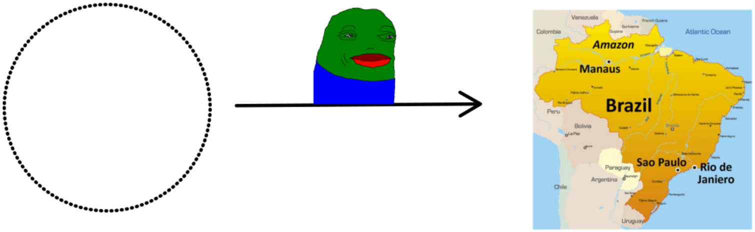
c. 1851: The Riemann Mapping Theorem

Let $U \subsetneq \mathbb{C}$ simply connected and open. Then, U is conformal (complex diffeomorphic) to $\mathbb{D}$.



c. 1907: The Uniformisation Theorem as Most People State it

Let Σ be a simply connected 1-dimensional complex manifold (a Riemann surface). Then, Σ is conformal to one of the following:

- $\mathbb{C}$, the entire complex plane (equipped with Euclidean geometry)
- $\mathbb{D}$, the unit disc (equipped with hyperbolic geometry)
- $\mathbb{C} \cup \{\infty\}$, the Riemann sphere (equipped with spherical geometry)

Natural Questions and Possible Generalisations

- Why is everything complex?
- Why do we only care about simply connected Riemann surfaces?
- Can we extend this to higher dimensions?

Outline

DIY: How to Make Your Manifold Simply Connected WITHOUT POWER TOOLS!

Take the *universal cover* of your topological space.

Going Backwards: Recovering the Base Space from the Universal Cover

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A 3D Analogue: Model Geometries

A model geometry is a (complete, homogenous), simply connected Riemannian manifold X together with its group of isometries $\operatorname{Isom}(X)$. A Riemannian manifold M is modelled by X if there exists an isometric diffeomorphism $M \cong X/\Gamma$ (where the action of Γ on X has no fixed points and is discrete).

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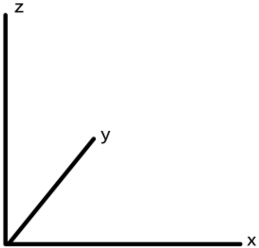
The Theorem? Conjecture? Whatever it is Statement

Every compact orientable 3-manifold can be decomposed into "canonical" submanifolds such that each piece is modeled by one of the eight Thurston geometries.

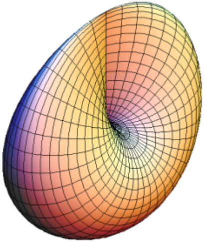
Three Familiar Thurston Geometries

- Euclidean geometry: $\mathbb{R}^3$ over $\mathbb{R}^3 \times O(3, \mathbb{R})$
- Spherical geometry: S^3 over $O(4, \mathbb{R})$
- Hyperbolic geometry: H^3 over $O^+(1, 3, \mathbb{R})$ (what)



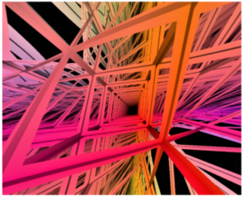
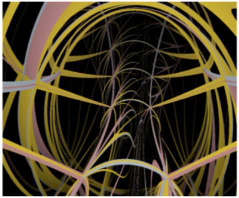
The Product Geometries

- $S^2 \times \mathbb{R}$
- $H^2 \times \mathbb{R}$



Uhhhhhhhhhhhhh

- $\mathrm{Nil} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$
- $\mathrm{Sol} = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$
- $\tilde{\mathrm{SL}}_2(\mathbb{R})$

An Important Corollary: The Poincaré Conjecture

A 3-manifold is homotopy equivalent to the 3-sphere if and only if it is homeomorphic to the 3-sphere.

Perelman and Hamilton's Legendary Work: Ricci Flows with Surgery

Analysed behaviour of solutions to

$$\frac{\partial}{\partial t} g_{ij}(t) = -2 \sum_{p,q} R_{ipjq} g^{pq}$$

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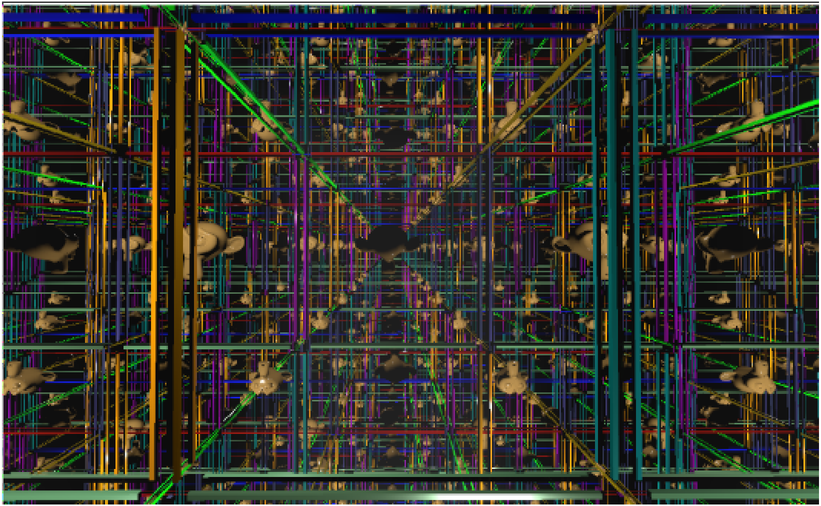
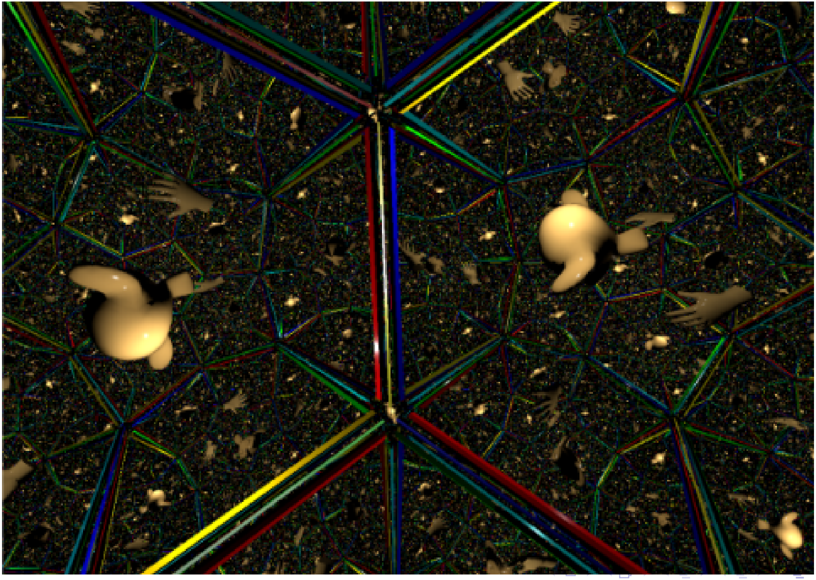
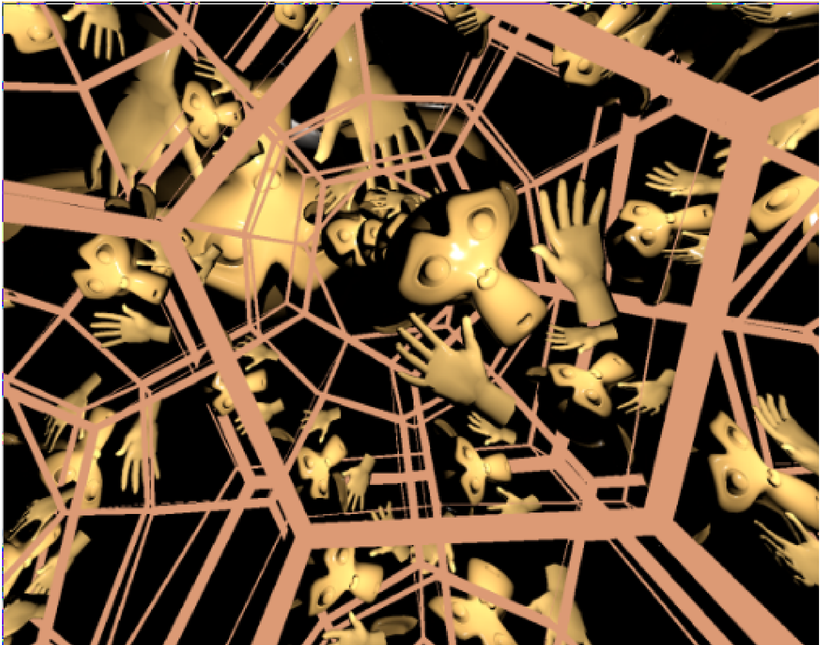
Monkeys in Inception (Available in the Director's Cut)



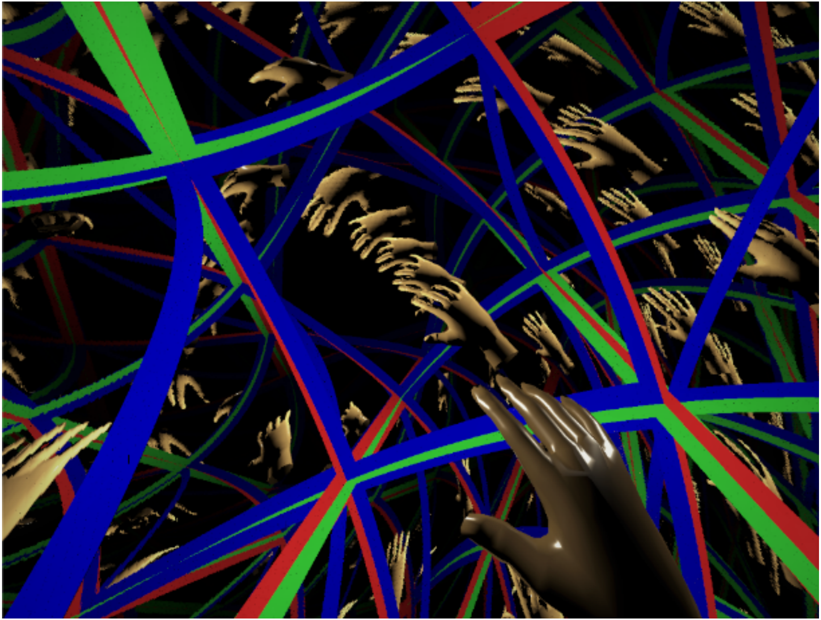
Monkeys in Siefert-Weber Dodecahedral Space, Modeled by H^3



Suzanne



Suzanne's Hands



References I

[1] Michael Boileau. *Geometrization of 3-Manifolds with Symmetries*. 2004.

[2] Ángel Prieto de la Cruz. "The Geometrisation Conjecture of 3-Manifolds". University of Barcelona, 2019.

[3] Jeremy Gray. "On the History of the Riemann Mapping Theorem". In: *Rendiconti del Circolo Matematico di Palermo* (1994), pp. 47–94.

[4] Tiago Novello et al. *How to see the eight Thurston geometries*. Sept. 2021.