Analytic Approaches to the First Hardy-Littlewood Conjecture and Related Problems

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Introduction

Theorem (The Prime Number Theorem)**:** $\pi(x) \sim \frac{x}{\log x}$*, where* $\pi(x)$ *is the prime counting function. Equivalently,* $\sum_{n \leq x} \Lambda(x) \sim x$*, where* Λ *is the von Mangoldt function.*

The prime number theorem needs no introduction. Perhaps the most intuitive interpretation of it is a probabilistic one, for the asymptotic on $\pi(x)$ in some sense describes the distribution of prime numbers among the integers. Importantly, sharp error terms in these asymptotics are not yet known, though they can be improved significantly by assuming the (generalised) Riemann hypothesis.

A natural thing to wonder is if this theorem can be used to prove things about patterns involving multiple primes, such as the twin prime conjecture. At first glance, the "density" of primes p less than x such that $p+2$ is also prime should be about $\frac{x}{\log^2 x}$: if p is a random integer between 1 and x, it has a $\frac{1}{\log x}$ chance of being prime, and likewise for $p+2$.

The issue, of course, is that the events of a random number p being prime and $p+2$ being prime are not independent; in fact, the fact that twin primes *aren't* asymptotic to $\frac{x}{\log^2 x}$ is verifiable numerically. Rather, there should be some corrective term or factor that adjusts the asymptotic in light of this probabilistic dependence. The Hardy-Littlewood conjecture does just that: it provides a concrete (though unproven) guess about what that corrective term should be.

To formalise this, we will let $k \geq 2$ and set $\mathcal{H} = (a_1, \ldots, a_k)$ for distinct integers a_1 through a_k. For prime numbers p, define the function $v_p(\mathcal{H})$ to be the number of distinct residue classes modulo p that the elements of $\mathcal{H}$ occupy. Finally, let $P(x; \mathcal{H})$ denote the number of integers $n \leq x$ such that $n + a_j$ is prime for all $j = 1, \ldots, k$. For $\mathcal{H} = (0, 2)$, for instance, $P(x; \mathcal{H})$ counts the number of twin primes below x.

Conjecture (Hardy-Littlewood)**:** $P(x; \mathcal{H}) \sim \mathfrak{G}(\mathcal{H})\, x \log^{-k} x$*, where*

$$\mathfrak{G}(\mathcal{H}) = \prod_{p \text{ prime}} \left(1 - \frac{v_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}.$$

Here, $\mathfrak{G}(\mathcal{H})$ is intuitively the "corrective factor" that adjusts for probabilistic dependence. Notice that if $v_p(\mathcal{H}) = p$ for any p, then $\mathfrak{G}(\mathcal{H}) = 0$; this reflects the fact that no matter what n is, one of the $n + a_j$'s will be congruent to 0 mod p, and thus $P(x; \mathcal{H})$ is finite for all x.

This probabilistic introduction, though helpful for visualising and rationalising this conjecture, is not entirely rigourous and is rather heuristic. In [1], Hardy and Littlewood take a rather different and frankly opaque approach that we will be describing in this paper. The probabilistic interpretation is not without merit, however, and we refer the reader to [2] and [3] for more developed discussions and results related to this perspective.

The Conjecture

This section closely follows section 56.2 (starting on page 54) in [1], which is the paper where Hardy and Littlewood first proposed and explained their conjecture.

When proving the prime number theorem, a common approach is to tackle the partial sums $\sum_{n\leq x} \Lambda(x)$ rather than directly tackling $\pi(x)$. In a similar vein, rather than directly tackling $P(x;\mathcal{H})$, Hardy and Littlewood tackled the sum

$$\sum_{n\leq x} \Lambda(n+a_1)\,\Lambda(n+a_2)\cdots\Lambda(n+a_k)\,.$$

This sum detects when $n+a_j$ are all prime powers rather than true primes. However, as is common in such approaches, the contribution from the non-prime prime powers is negligible compared to those of the true primes.

Sums of the form $\sum_{n\leq x}\Lambda(n)$ and $\sum_{n\leq x}\Lambda(n)\,\chi(n)$ for some character χ are far more well-understood, and the idea is to cleverly split the more complicated sum into simpler, more manageable sums. This is done using a circle method-esque argument: the sum of interest is related to the coefficients of a Fourier series, and the Fourier series is hopefully more manageable than the original sum. For brevity's sake, we will assume some basic familiarity with the circle method.

First, we introduce the modified sum

$$f_{\mathcal{H}}(z) = \sum_{n\leq x} \Lambda(n+a_1)\cdots\Lambda(n+a_k)\,z^n,$$

where z is a complex parameter. When $|z|<1$, the sum converges, but when $|z|>1$, the sum diverges as $x\to\infty$. We are interested in the behaviour of $f_{\mathcal{H}}$ as $|z|$ gets close to 1. Of note is when z is a root of unity: the z^n term then acts as a character (or sum of several characters). More broadly, letting $z=\exp(2\pi i\alpha)$ for a real parameter α turns the sum into a Fourier series.

To these ends, let p_0 and q_0 be coprime positive integers.

$$r^{a_k} f_{\mathcal{H}}\left(r^2 e\left(\frac{p_0}{q_0}\right)\right) = \sum_{n\leq x} \Lambda(n+a_1)\cdots\Lambda(n+a_k)\,r^{2n+a_k} e\left(\frac{np_0}{q_0}\right),$$

where $e\left(\frac{p_0}{q_0}\right) = \exp\left(2\pi i \frac{p_0}{q_0}\right)$. Serendipitously notice that this is exactly the integral

$$= \int_0^1 \left(\sum_{n \le x} \Lambda(n + a_1) \cdots \Lambda(n + a_{k-1}) r^n e\left(n\varphi + \frac{np_0}{q_0}\right) \right) \times$$
$$\left(\sum_{m \le x} \Lambda(m + a_k) r^{m+a_k} e\left(-(m + a_k)\varphi\right) \right) e(a_k \varphi) \, \mathrm{d}\varphi.$$

Upon distributing the two sums, the summand becomes a product of k Λ's, though the last term is no longer "aligned" with the first $k-1$ terms from the first sum. This is where the exponentials come in: when the terms are combined, one has a factor of $e\left((n-m)\varphi\right)$ that depends on φ. After integrating, any terms where $n \neq m$ vanish, leaving precisely the sum from before. Note that we may swap the sums and the integral freely. The two sums in the integrand closely resemble $f_{\mathcal{H}}$: shifting the second sum yields

$$= \int_0^1 f_{\mathcal{H} \setminus \{a_k\}} \left(re\left(\varphi + \frac{p_0}{q_0}\right) \right) f_{(a_k)}\left(re(-\varphi)\right) e(a_k \varphi) \, \mathrm{d}\varphi.$$

This manipulation has allowed us to express $f_{\mathcal{H}}$ in terms of simpler sums. In particular, $f_{(a_k)}\left(re(-\varphi)\right) = \sum_{n \le x} \Lambda(n) r^n e(-n\varphi)$, and when φ is close to a rational number, $e(-n\varphi)$ is suspiciously manageable.

The following treatment of this approximation is based on chapter 26 of [4] and in [5], though we deviate slightly in accounting for the extra factor of r^n. These two sources specifically discuss Vinogradov's theorem regarding sums of three primes, wherein the sum $\sum_{n \le x} \Lambda(n) e(-n\varphi)$ plays a large role. Many of the bounds and asymptotics for this sum are applicable in this setting, too.

Write $\mathfrak{M}(q,p)$ as a real interval of unspecified width (to be made arbitrarily small) centred at $\frac{p}{q}$, where p and q are relatively prime. These intervals are taken modulo 1 so that each $\mathfrak{M}(q,p) \subseteq [0,1]$. Notably, the widths of these intervals can be specified uniformly for all $p, q \le P$ for some (similarly unspecified) number P. These bounds and widths are chosen so that the union of all $\mathfrak{M}(q,p)$'s is not all of $[0,1]$, and so that each $\mathfrak{M}(q,p)$ is disjoint. These constitute the "major arcs".

For $\varphi \in \mathfrak{M}(q,p)$, write $\varphi = \frac{p}{q} + \theta$ for θ very small compared to $1 - r$. Then,

$$f_{(a_k)}\left(re(-\varphi)\right) = \sum_{n \le x} \Lambda(n) e\left(-\frac{np}{q}\right) r^n e(-n\theta).$$

The idea is to rewrite $e\left(-\frac{np}{q}\right)$ in terms of Gauss sums and characters modulo q. The contribution from terms with $(n,q) > 1$ is $O\left(\log^2 x\right)$; when $(n,q) = 1$, we have that

$$\frac{1}{\phi(q)} \sum_{\chi} \tau(\bar{\chi}) \chi(-np) = e\left(-\frac{np}{q}\right),$$

where χ ranges over all characters mod q and $\tau(\bar{\chi})$ is the Gauss sum $\sum_{a\in\mathbb{Z}/q\mathbb{Z}^\times} \bar{\chi}(a)\, e\left(\frac{a}{q}\right)$. Inserting this into the big expression yields

$$f_{(a_k)}\left(re\left(-\varphi\right)\right) = \frac{1}{\phi(q)} \sum_{\chi} \tau(\bar{\chi})\, \chi(p) \sum_{n\leq x}{}' \chi(-n)\, \Lambda(n)\, r^n e(-n\theta) + O\left(\log^2 x\right).$$

Here, the $\sum'$ indicates that n ranges over integers prime to q. We handle the trivial character χ_0 and the nontrivial characters separately. Let $N = \lfloor x \rfloor$. Apply summation by parts to the inner sum:

$$\sum_{n\leq x}{}' \Lambda(n)\, \chi(-n)\, r^n e(-n\theta) = r^N e(-N\theta)\, \psi(x, \bar{\chi}) - (\log r - 2\pi i\theta) \int_1^N r^t e(-t\theta)\, \psi(t, \bar{\chi})\, \mathrm{d}t.$$

Here, we used $\chi(-n) = \bar{\chi}$ and the Chebyshev function $\psi(x, \chi) = \sum_{m\leq x} \Lambda(m)\, \chi(m)$. Using standard estimates on ψ when $\chi \neq \chi_0$, we obtain the bound

$$= O\left(\left(1 + \left|\log r - \theta\right| N\right) N \exp\left(-c\sqrt{\log N}\right)\right),$$

where c is a constant depending only on N and q. We refer the reader to [4] §19, §21, and §22 for more details on formulae and bounds on ψ. Write $T(\theta) = \sum_{n\leq x} r^n e(-n\theta)$, a geometric series; we have that

$$\sum_{n\leq x}{}' \Lambda(n)\, \chi_0(-n)\, r^n e(-n\theta) = T(\theta) + O\left(\left(1 + \left|\log r - \theta\right| N\right) N \exp\left(-c\sqrt{\log N}\right)\right),$$

with the gritty details left as an exercise. Just kidding, please see [4] or [5] for the full argument. The key is that the χ_0 term in the outer sum dominates the other terms; using $\tau(\chi_0) = \mu(q)$ (where μ is the Möbius function) and $\left|\tau(\chi)\right| \leq \sqrt{q}$ otherwise, we arrive at the bound

$$f_{(a_k)}\left(re\left(-\varphi\right)\right) = \frac{\mu(q)}{\phi(q)} T(\theta) + O\left(\left(1 + \left|\log r - \theta\right| N\right) N \sqrt{q} \exp\left(-c\sqrt{\log N}\right)\right).$$

Finally, using the fact that r is close to 1 and θ is close to zero allows one to simplify this bound and rewrite $T(\theta) = \frac{1-r^N e(-N\theta)}{1-re(-\theta)} \sim \frac{1}{1-r}$ so that

$$f_{(a_k)} \sim \frac{\mu(q)}{\phi(q)} \cdot \frac{1}{1-r},$$

with an error term of the form $O\left(N \exp\left(-c'\sqrt{\log N}\right)\right)$, where c' is some constant depending only on N and q.

It is at this point that Hardy and Littlewood (somewhat implicitly) posed two critical assumptions. First, we assume that the contribution of

$$\int_0^1 f_{\mathcal{H}\setminus\{a_k\}}\left(re\left(\varphi + \frac{p_0}{q_0}\right)\right) f_{(a_k)}\left(re\left(-\varphi\right)\right) e\left(a_k\varphi\right) \mathrm{d}\varphi$$

is small outside the major arcs. This allows us to relate the asymptotic of $f_{\mathcal{H}}$ to the asymptotic along the major arcs, which we described above. Second, they assumed that asymptotics for $f_{\mathcal{H}}$ existed to begin with, and they labeled

$$f_{\mathcal{H}}\left(re\left(\alpha\right)\right) \sim \frac{g_{\mathcal{H}}\left(\alpha\right)}{1-r}$$

for some function $g_{\mathcal{H}}$ depending on the real parameter α. Notably, $g_{(a_k)}\left(\frac{p}{q}+\theta\right)$ for $\theta \ll 1-r$ is precisely $\frac{\mu(q)}{\phi(q)}$, as determined above.

By using the integral relationship and using the asymptotics assumed and/or derived on the major arcs, one arrives at the following:

$$\begin{aligned} r^{a_k} f_{\mathcal{H}}\left(r^2 e\left(\frac{p_0}{q_0}\right)\right) &= \int_0^1 f_{\mathcal{H}\setminus\{a_k\}}\left(re\left(\frac{p_0}{q_0}+\varphi\right)\right) f_{(a_k)}\left(re\left(-\varphi\right)\right) e\left(a_k\varphi\right) \mathrm{d}\varphi \\ &\sim \sum_{p,q} \frac{\mu\left(q\right)}{\phi\left(q\right)} \cdot g_{\mathcal{H}\setminus\{a_k\}}\left(\frac{p}{q}+\frac{p_0}{q_0}\right) e\left(\frac{a_k p}{q}\right) \int_{\mathfrak{M}(q,p)} \frac{\mathrm{d}\theta}{\left(1-re\left(\theta\right)\right)\left(1-re\left(-\theta\right)\right)} \\ &\sim \frac{1}{1-r^2} \sum_{p,q} \frac{\mu\left(q\right)}{\phi\left(q\right)} \cdot g_{\mathcal{H}\setminus a_k}\left(\frac{p}{q}+\frac{p_0}{q_0}\right) e\left(\frac{a_k p}{q}\right) \end{aligned}$$

Here, the sum ranges over all p, q within the major arcs. Since r is "very close" to 1, we may ignore the r^{a_k} term in the front. In addition, we had assumed that $f_{\mathcal{H}}\left(r^2 e\left(\frac{p_0}{q_0}\right)\right) \sim g_{\mathcal{H}}\left(\frac{p_0}{q_0}\right) \cdot \frac{1}{1-r^2}$, and this produces a recurrence relation between the g's. When $p_0 = 0$, i.e. when we're observing the behaviour of $f_{\mathcal{H}}\left(r^2\right)$, one gets the relationship

$$g_{\mathcal{H}}\left(0\right) \sim \sum_{p,q} \frac{\mu\left(q\right)}{\phi\left(q\right)} \cdot g_{\mathcal{H}\setminus\{a_k\}}\left(\frac{p}{q}\right) e\left(\frac{a_k p}{q}\right).$$

Hardy and Littlewood then examined this recurrence relation and solved it to produce the asymptotic proposed in their conjecture; for these computations, we refer the reader to [1] (see pg. 55 and 56).

Author's note: reading [1] was difficult, and Hardy and Littlewood provided few explanations and computations for certain bounds and asymptotics, particularly those involving $f_{(a_k)}$. We have reverse-engineered these steps by adapting Vinogradov's arguments and bounds, though we make no guarantees for their validity.

The Technique in Practise

Three main difficulties arise when applying these techniques and arguments to concrete problems. First, justifying that the contribution to the integral from the minor arcs does not dominate the error term in the major arcs is often difficult, and this point was assumed by

Hardy and Littlewood in [1]. Second, ensuring that the error terms in the major arcs themselves don't dominate the main asymptotic can be especially difficult, even when assuming the Riemann Hypothesis (which affords sharper bounds on ψ). Third, and most importantly, many results only specify *upper bounds* on these integrals. In certain applications (as we shall soon see), these upper bounds are enough; however, they do not guarantee a precise asymptotic like what Hardy and Littlewood conjectured.

It is tempting to apply a similar argument or perform similar manipulations when k is very small, say $k = 2$. This means we only need to split the sum once. Consider $\mathcal{H} = (0, 2)$, which corresponds to the twin prime problem.

$$\sum_{n \leq x} \Lambda(n) \Lambda(n+2) = \int_0^1 \left(\sum_{n \leq x} \Lambda(n) e(n\varphi) \right) \left(\sum_{m \leq x} \Lambda(m+2) e\big(-(m+2)\varphi\big) \right) e(2\varphi) \, \mathrm{d}\varphi.$$

Writing $S(\varphi) = \sum_{n \leq x} \Lambda(n) e(n\varphi)$, we may rewrite this as

$$\sum_{n \leq x} \Lambda(n) \Lambda(n+2) = \int_0^1 S(\varphi) S(-\varphi) e(2\varphi) \, \mathrm{d}\varphi.$$

Write $\mathfrak{m}$ for the minor arcs; naïvely applying the triangle inequality yields an upper bound of order $x \log x$ for $\int_{\mathfrak{m}} \big| S(\varphi) S(-\varphi) \big| \, \mathrm{d}\varphi$: this is because $\sum_{k \leq x} \Lambda(x)^2 \sim x \log x$. However, the asymptotic we want for the sum is on the order of $x \log^{-2} x$, which is somewhat off. One may attempt to improve this bound by instead trying

$$\left| \int_{\mathfrak{m}} S(\varphi) S(-\varphi) e(2\varphi) \, \mathrm{d}\varphi \right| \leq \sup_{\mathfrak{m}} \big| S(\varphi) \big| \int_{\mathfrak{m}} \big| S(-\varphi) \big| \, \mathrm{d}\varphi,$$

where the sup is upper-bounded by $x \log^{-A} x$ for some constant A (see [4], §25) while the integral is bounded by x. However, the resultant upper bound is still $x^2 \log^{-A} x$, which is actually worse than the naïve bound. In general, it appears that the best upper bounds for these integrals for an arbitrary k-tuple with $k > 1$ will be x^{k-1}, possibly with a nice logarithmic factor; unfortunately, these will dominate the asymptotic of $x \log^{-k} x$.

Let us now bring attention to an application of this technique to a significantly less restrained problem involving primes. Specifically, we consider the ternary Goldbach conjecture, which states that every odd integer bigger than 5 is the sum of three primes. Define the function

$$r(N) = \sum_{k_1 + k_2 + k_3 = N} \Lambda(k_1) \Lambda(k_2) \Lambda(k_3).$$

This "detects" prime powers k_1, k_2, k_3 whose sum is N; in the limit as $N \to \infty$, the contribution from composites becomes negligible. Hardy and Littlewood were able to deduce the follow theorem of Vinogradov by assuming the Riemann hypothesis, but the theorem is named after Vinogradov because he was able to prove it without such a huge assumption.

Theorem (Vinogradov's Theorem (one of them))**:** *For all $A > 0$,*

$$r(N) = \frac{\mathfrak{S}(N) N^2}{2} + O\left(N^2 \log^{-A} N\right),$$

where $\mathfrak{S}(N)$ *is given by*

$$\mathfrak{S}(N) = \prod_{p|N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\nmid N}\left(1 + \frac{1}{(p-1)^3}\right).$$

Note that this $\mathfrak{S}(N)$ is different from $\mathfrak{S}(\mathcal{H})$ defined earlier; please forgive the author, as both are standard notation. Importantly, $\mathfrak{S}(N)$ is bounded both above and below by constants, meaning that when N is sufficiently large, $r(N)$ is certainly positive. That is to say, every sufficiently large N is expressible as the sum of three primes.

The argument that Vinogradov gave utilised the Hardy-Littlewood circle method. Write $S(\varphi) = \sum_{n\leq N}\Lambda(n)\, e(n\varphi)$ for real φ, and notice that

$$r(N) = \int_0^1 S(\varphi)^3\, e(-N\varphi)\, \mathrm{d}\varphi.$$

This is because upon expanding the cubed series in the integrand, one sees that

$$S(\varphi)^3\, e(-N\varphi) = \sum_{k_1,k_2,k_3\leq N}\Lambda(k_1)\Lambda(k_2)\Lambda(k_3)\, e\big((k_1+k_2+k_3-N)\,\varphi\big),$$

and the only terms without an exponential are those where $k_1+k_2+k_3 = N$. Treating the major arcs for this integral is more or less identical to that in the previous section; when treating the minor arcs, one bounds

$$\sup_{\mathfrak{m}}\big|S(\varphi)\big|\int_{\mathfrak{m}}\big|S(\varphi)\big|^2\,\mathrm{d}\varphi.$$

The sup produces a bound of $x\log^{-A}x$ while the integral produces a bound of $x\log x$; these together form the desired asymptotic.

Soon after, van der Corput [6] was able to slightly modify the argument to produce the following result:

Corollary: *There exist infinitely many prime triplets* p_1, p_2, p_3 *lying in a nontrivial arithmetic progression.*

The manuscript is in German, and the author is unable to find a translation. However, it appears that appealing to the integral

$$\int_0^1 S(\varphi)^2\, S(-2\varphi)\,\mathrm{d}\varphi$$

could produce this result: expanding the sum picks out prime powers k_1, k_2, k_3 such that $k_1 + k_2 = 2k_3$, i.e. that the three numbers lie in arithmetic progression. This integral can be bounded with the same techniques that Vinogradov used.

In the 1980s, Heath-Brown [7] gave a generaliation as follows:

Theorem: *There exist infinitely many primes p_1, p_2, p_3 and an almost-prime p_4 having at most two prime factors all lying in a nontrivial arithmetic progression.*

When trying to get four primes in progression, the natural thing to do is to write

$$\sum_{\substack{k_1,k_2,k_3,k_4 \le N \\ k_1+k_3=2k_2 \\ k_2+k_4=2k_3}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3)\Lambda(k_4) = \int_0^1 \int_0^1 S(\alpha)\, S(-2\alpha+\beta)\, S(\alpha-2\beta)\, S(\beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta.$$

Upon expanding the integrand, the condition that $k_1 + k_3 = 2k_2$ is captured by the cancellation of exponentials involving α; similarly, the condition that $k_2 + k_4 = 2k_3$ is captured by the cancellation of exponentials in β. However, applying the bounds Vinogradov gave is fruitless, for it appears to (at best) give an error term on the order of $N^3 \log^{-A} N$ while the sum should be bounded by $\frac{1}{2}\mathfrak{G}(N)\, N^2$. Even when assuming the Riemann hypothesis, these error terms still remain comparable to the sum itself.

Instead, Heath-Brown relaxed these constraints further by allowing k_1 or k_4 to have at most 2 prime factors. This was done by using standard linear sieve methods, but those arguments are beyond the scope of this paper. Combining such methods with the circle method gave Heath-Brown the right amount of refinement on the error terms.

Adjacent and Related Results

The first thing to note is that the Hardy-Littlewood conjecture is still very open. We shall detail some statements that bring us closer to the conjecture; however, none of these results provide concrete asymptotics the way Hardy and Littlewood did, and they all are considerably weaker than their conjecture. Nevertheless, the progress that has been made is somewhat astounding, and they are worth mentioning and appreciating.

First, Vinogradov did not specify what “sufficiently large” meant. By closely following his arguments, a numerical limit of $N > 10^{1370}$ was determined as “sufficiently large”, and since the ternary Goldbach conjecture had, at some point, been verified for all integers up to 10^{20}, the only obstruction to this conjecture was computational power. In 2013, Helfgott [8] used sieve methods to refine the bound on minor arcs, thereby reducing the computational load required, ultimately resulting in a (currently unverified?) proof of the ternary Goldbach conjecture.

Around the same time Helfgott announced his proof, Yitang Zhang [9] proved the astounding result that $\liminf_{n\to\infty} (p_{n+1} - p_n) < \infty$, where p_n is the n-th prime number. He produced the explicit bound of 7×10^7, but this has subsequently been sharpened to 246 or lower since. This result followed a similar result from Goldston, Pintz, and Yıldırım [10], who proved the weaker form

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0$$

in 2005. This result was strengthened by the same authors not long after, though not to the extent that Zhang did.

And finally, in 2008, Ben Green and Terence Tao [11] proved that the primes contain *arbitrarily long* arithmetic progressions, which is now known as the Green-Tao theorem. The closest result to this was Heath-Brown's result on progressions of 3 primes and a fourth "almost-prime", though none of these results provide an asymptotic like the one conjectured by Hardy and Littlewood.

Hardy, Littlewood, Ramanujan, and Vinogradov were all incredibly prolific and active around 100 years ago; the conjectures and questions they tackled were older yet. It seems that despite the astounding progress within the past half-century, their conjectures remain wide open and unapproachable by modern techniques.

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