The Ramanujan-Petersson Conjecture

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Introduction

One of the first examples of a nontrivial cusp form (on $\mathrm{SL}_2(\mathbb{Z})$) is the function $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$, a cusp form of weight 12. Ramanujan studied this function, specifically the Fourier coefficients $\Delta = \sum_{n=1}^{\infty}\tau(n)q^n$. By computing the first 30 or so values of $\tau(n)$ and finding patterns, Ramanujan proposed the following conjecture:

Conjecture: *For all n, m coprime, $\tau(nm) = \tau(n)\tau(m)$ (i.e., τ is multiplicative). In addition, for all primes p and positive integers n, one has $\tau\left(p^{n+1}\right) = \tau\left(p^n\right)\tau(p) - p^{11}\tau\left(p^{n-1}\right)$ and $|\tau(p)| \leq 2p^{\frac{11}{2}}$.*

The first two conditions are equivalent to producing a factorisation of the L-function associated to Δ, where one gets

$$L(\Delta, s) = \sum_{n=1}^{\infty}\tau(n)n^{-s} = \prod_{p}\left(1-\tau(p)p^{-s}+p^{11-2s}\right)^{-1},$$

where the product ranges over all primes p.

An astute reader can perhaps point out that these two conditions are uncannily similar to the relationships of Hecke operators; indeed, generalising this conjecture was one of the motivations for Hecke's work. This generalisation, also named after Petersson, can be stated as follows:

Conjecture: *Let $f = \sum_{n=1}^{\infty}a_n(f)q^n \in \mathcal{S}_k\left(\Gamma_1(N)\right)$ be a normalised Hecke eigenform. Then $|a_n(f)| \leq 2p^{\frac{k-1}{2}}$.*

Yet again, one may consider this conjecture to be a statement about the L-functions $L(f, s)$, and one mimicks the manipulations used to produce an Euler product for $L(\Delta, s)$. Mordell proved the Euler factorisation early in the twentieth century, but the bound on Fourier coefficients remained unsolved until Deligne proved the Weil conjectures. The Ramanujan-Petersson conjecture (for weights $k \geq 3$) followed as a corollary; the weights $k = 1, 2$ cases were handled separated, but they have been solved.

However, the theory of L-functions has since expanded far beyond the scope of modular forms; with it, the Ramanujan-Petersson conjecture has generalised beyond its original statement. Some generalisations have been demonstrated to be false, others have been solved, but many broad cases have remained completely conjectural and unsolved.

The goal of the following pages is thus to provide a description of the larger context in which modular forms fit. We aim to provide an expositional level of detail while hopefully offering a broad and somewhat general perspective on the Ramanujan-Petersson conjecture. This conjecture is of somewhat surprising relevance and interest to modern mathematics, and we hope to illuminate its importance and development.

Maass Forms and Number Fields

Hecke Characters, Hecke L-functions, and Artin L-functions

Let K be a number field, and let $\mathcal{O}_K$ its ring of integers. Let $\mathfrak{m}$ be an integral ideal of K. Let $J^{\mathfrak{m}}$ be the ideals of K relatively prime to $\mathfrak{m}$.

Definition: *A Hecke character (or Grössencharaktere) modulo* $\mathfrak{m}$ *is a homomorphism* $\chi : J^{\mathfrak{m}} \to S^1 \subset \mathbb{C}$ *of the form* $\chi = \chi_1 \chi_\infty$*, where* $\chi_1 : (\mathcal{O}/\mathfrak{m})^\times \to S^1$ *and* $\chi_\infty : \mathbb{R}^\times \to S^1$ *are both characters.*

If χ' *is another Hecke character mod* $\mathfrak{m}'$ *for some* $\mathfrak{m}' \mid \mathfrak{m}$*, so that* $\chi' = \chi$ *on* $J^{\mathfrak{m}}$*,* χ *is said to be a restriction of* χ'*. The conductor of* χ *is the smallest divisor* $\mathfrak{q} \mid \mathfrak{m}$ *such that* χ *is the restriction of another Hecke character mod* $\mathfrak{q}$*.*

We remark here that Hecke characters are often introduced as characters of the idèle class group, or continuous homomorphisms $\chi : K^\times \backslash \mathbb{A}_K^\times \to S^1$. This is a much more concise definition that relies on more modern terminology than was available to Hecke.

Given a Hecke character χ, one can extend it to the integral ideals of K by setting $\chi(\mathfrak{a}) = 0$ when $\mathfrak{a}$ and $\mathfrak{m}$ are not coprime. This allows us to define the Hecke L-function

$$L(\chi, s) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \, (N\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p}) \, (N\mathfrak{p})^{-s}\right)^{-1},$$

where N is the absolute norm, the sum ranges over integral ideals $\mathfrak{a}$, and the Euler product ranges over prime ideals $\mathfrak{p}$.

One may continue to define the completed Hecke L-series

$$\Lambda(\chi, s) = (|d_K| \, N\mathfrak{m})^{\frac{s}{2}} \, L_\infty(\chi, s) L(\chi, s),$$

where d_K is the discriminant of K over $\mathbb{Q}$ and L_∞ is another L function. We refer the reader to [1] for a more complete discussion and development of the theory of Hecke L-functions.

There are several things to note about Hecke L-functions and Hecke characters. First, when χ is a Hecke character and has finite order, it is actually exactly a Dirichlet character; hence Hecke L-functions are a broad generalisation of Dirichlet L-functions that incorporate the arithmetic of a number field. Second, the completed Hecke L-function admits a meromorphic continuation to $\mathbb{C}$ (it can be shown to converge absolutely for $\operatorname{Re} s \gg 0$), and it satisfies the functional equation

$$\Lambda(\chi, s) = W(\chi)\Lambda(\overline{\chi}, 1-s)$$

for some $W(\chi) \in S^1$ depending only on χ. One may observe that $L(\chi, s)$ contains the data of the non-archimedean places of K; the completed Λ has an additional factor of L_∞ that can be thought to include the archimedean places. The meromorphic continuation thus allows for an analytic approach to understanding the arithemetic of K.

Artin later noticed that Hecke characters χ lay in a correspondence with two-dimensional Galois representations $\rho_\chi : G_K \to \mathrm{GL}_2(\mathbb{C})$, where G_K is the absolute Galois group of K. This motivated a generalisation of Hecke's L-functions called Artin L-functions.

Let L/K a finite Galois extension with Galois group G, and let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation. Artin defined an incomplete L-function by considering

$$L(\rho, s) = \prod_{\mathfrak{p}} \det\left(I_n - \rho\left(\mathrm{Frob}_{\mathfrak{p}}\right)\left(N\mathfrak{p}\right)^{-s}\right)^{-1},$$

where the product is taken over unramified primes $\mathfrak{p}$ and $\mathrm{Frob}_{\mathfrak{p}}$ is any Frobenius element over $\mathfrak{p}$ (in fact, it is independent of this choice).

Artin then "completed" the L-function to $\Lambda(\rho, s)$ by extending the above product to include the ramified primes, then adding some factors involving the gamma function Γ for the archemidean primes. This again produces a functional equation relating $\Lambda(\rho, s) = W(\rho)\Lambda\left(\overline{\rho}, 1-s\right)$, where again $W(\rho) \in S^1$ depends only on ρ. For a deeper discussion of Artin L-functions with a historical perspective, see [2].

While these Artin L-functions encode a wealth of arithmetic information, it is difficult to obtain much information about their analytic properties. Using the convergence of $\Lambda(\rho, s)$ for $\mathrm{Re}\, s > 1$ together with the functional equation, Artin was able to establish a meromorphic continuation to $\mathbb{C}$; however, recognising when this is an *analytic* continuation for nontrivial ρ is still an open question.

Maass Forms and Conjectural "Modularity" of Galois Representations

Perhaps the above discussion hints at a motivation to connect these highly arithmetic objects, Artin L-functions, to an object whose analytic properties we can better understand. Such a correspondence is highly conjectural and remains open.

As a related example, consider the recently proven modularity theorem, which says that L-function constructed from elliptic curves over $\mathbb{Q}$ are realised as $L(f, s)$ for some modular form f. Given a Galois representation ρ as above, does there exist a modular form (or related object) f whose L-function coincides with $L(\rho, s)$?

Hecke considered the case when K was an imaginary quadratic field, for which he constructed a holomorphic Maass form (to be defined later) of weight k (depending on χ_∞) and level $|d_K| \cdot N\mathfrak{q}$, where again $\mathfrak{q}$ is the conductor of χ. This form is given by

$$f_\chi = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \left(N\mathfrak{a}\right)^{\frac{k-1}{2}} q^{N\mathfrak{a}}.$$

Maass adapted Hecke's construction to real quadratic number fields, but found that this generalisation lost its holomorphicity. This motivated him to introduce the space of Maass forms, which serve as a class of functions that relax the conditions on modular forms and, to some extent, aim to answer this version of modularity.

Maass forms are defined as functions on $\mathbb{H}$ that transform well under the action of a congruence subgroup Γ, that are L^2-integrable with respect to the hyperbolic measure $y^{-2}dxdy$ on the quotient $\Gamma\backslash\mathbb{H}$, that satisfy a moderate growth condition near the cusps (analogous to meromorphicity), and are eigenfunctions of the weight k non-Euclidean Laplacian $\Delta_k = -y^2\left(\partial_x^2 + \partial_y^2\right) + iky\partial_x$. For a more precise definition of Maass forms, see [3]. We also refer the reader to section 4 of [4], which has a concise but detailed exposition of Maass forms.

If f is a modular form of weight k, then $y^{\frac{k}{2}}f$ is a Maass form of weight k. Thus Maass forms are a generalisation of the space of modular forms after this normalisation; in addition to the weight k Laplacian, one may define a generalisation of Hecke operators T_n to the space of Maass forms, and in fact, these T_n's commute with Δ_k. The classical notions of cusp forms, newforms, and oldforms carry over from the setting of modular forms. A Maass form that is an eigenfunction of each T_n is called a Hecke-Maass form.

It is under this more general setting that Maass was able to construct analytic objects whose L-functions coincided with the Hecke L-function of a real quadratic field. Moreover, for certain irreducible 2-dimensional complex representations of $G_{\mathbb{Q}}$ (the absolute Galois group of $\mathbb{Q}$), Langlands and Tunnell demonstrated the existence of Maass forms that shared an L-function with the representation. See [3] for a full description of these cases.

Conjectures of Ramanujan-Petersson and Selberg

Upon considering Maass forms as a broadening of modular forms, it is natural to ask about an analogue of the Ramanujan-Petersson conjecture. There are several different statements of the conjecture, each carrying a different perspective. On one hand,

Conjecture: *Let f be a Hecke-Maass form of weight k, and let $a_n(f)$ be its nth Fourier coefficient. Then $|a_n(f)| \leq 2p^{\frac{k-1}{2}}$.*

On the other hand, it is worthwile to consider T_n to be a linear operator acting on the Hilbert space $L^2\left(Y(N)\right)$ (or $Y_1(N)$, or $Y_0(N)$), which can bee seen as the space of Γ-invariant functions on $\mathbb{H}$ whose L^2 norm is finite. These T_n's turn out to be self-adjoint, and the Ramanujan-Petersson conjecture considers the restriction of T_n to the subspace $L_0^2(Y(N))$ of Maass cusp forms of weight 0. There, it states that the operator norm of this restriction is at most 2, or more precisely that $\|T_p\|_{L^2(Y(N))} \leq 2$ for all primes p.

This establishes a link to the tools of functional analysis and spectral theory, but the picture is yet incomplete. The Hecke operators also commute with Δ_k, which for $k = 0$ is the usual non-Euclidean Laplacian operator Δ, which also acts on $L^2(Y(N))$. Selberg was interested in the spectral properties of Δ, which has an interesting spectrum on this larger space of functions. On one hand, it has a continuous spectrum spanned by non-holomorphic Eisenstein series, and the eigenvalues taken all fall in the range $\left[\frac{1}{4}, \infty\right)$. More interestingly, however, they also have a discrete spectrum consisting of Maass forms, which are more mysterious.

Traditionally, the eigenvalues from the discrete spectrum are expressed as $\frac{1-s^2}{4}$ for a complex parameter s. Since Δ is a real positive-definite symmetric operator, it follows that its eigenvalues too must be real and positive. Selberg conjectured that the parameters s were all purely imaginary, hence forcing the discrete spectrum to have eigenvalues at least $\frac{1}{4}$. This is Selberg's eigenvalue conjecture. For a good expository article on Selberg's conjecture, see [5].

The T_p's are suggestive of non-archimedean places of $\mathbb{Q}$, and Δ could perhaps fit in as the archimedean place of $\mathbb{Q}$. Is there a formulation of this conjecture that unifies the two?

The answer is yes, and mathematicians such as Satake and Langlands realised this when they rephrased Maass forms in the context of automorphic representations. The limited intellectual capacity of the author means that (like most things in this exposition) our discussion will be minimal and simplified, but we will aim to provide some background nonetheless for completeness.

We will assume familiarity with adéles, and we refer the reader to [6] for a more complete development of the necessary background and ensuing theory. Let K be a number field, $\mathbb{A}$ its ring of adéles, and $\mathbb{A}_{\text{fin}}$ the finite adéles. Under the right topology, $\mathrm{GL}(n, \mathbb{A})$ is a unimodular locally compact topological group, so it admits a (left and right) Haar measure μ. $\mathrm{GL}(n, K)$ is then a discrete subgroup of $\mathrm{GL}(n, \mathbb{A})$. Let $Z(n, \mathbb{A})$ denote the scalar $n \times n$ matrices with entries in $\mathbb{A}$. The quotient $Z(n, A)\,\mathrm{GL}(n, K)\backslash \mathrm{GL}(n, \mathbb{A})$ has finite volume with respect to its Haar measure.

Let $\omega : \mathbb{A}^\times / K^\times \to S^1$ be a character (an adélisation of the earlier Hecke character), and define the space $L^2(\mathrm{GL}(n, K) \setminus \mathrm{GL}(n, \mathbb{A}), \omega)$ to be the space of all square-integrable functions f on the quotient such that $f(zg) = \omega(z) f(g)$ for all $z \in Z(n, \mathbb{A})$. Define $\mathcal{A}(\mathrm{GL}(n, K)\backslash \mathrm{GL}(n, \mathbb{A}))$ to be the space of automorphic forms, consisting of functions $f \in L^2(\mathrm{GL}(n, K) \setminus \mathrm{GL}(n, \mathbb{A}))$ satisfying some growth, smoothness, and finiteness conditions. Automorphic representations of $\mathrm{GL}(n, \mathbb{A})$ are (roughly speaking) those induced by representations of subquotients of $\mathcal{A}(\mathrm{GL}(n, K)\backslash GL(n, \mathbb{A}))$.

Every irreducible representation π of $\mathrm{GL}(n, \mathbb{A})$ satisfying some *admissability* conditions can be identified with $\bigotimes_v \pi_v$, where each π_v is a representation of $\mathrm{GL}(n, K_v)$ for places v of K, and all but finitely many of the π_v's are unramified. Automorphic representations described prior naturally correspond to these tensored representations.

The Ramanujan-Petersson conjectures and the Selberg conjectures are magically unified under this new guise. From Maass forms, one may construct an irreducible (admissable) representation of $\mathrm{GL}(n, \mathbb{A})$, and the conjectures of the behaviour T_p translate into the *temperedness* of the components π_v for finite places v. At the same time, Selberg's conjecture on the eigenvalues of Δ translate to temperedness of the infinite components of π.

This is the generalised Ramanujan-Petersson conjecture. The above discussion of automorphic representations can be generalised to certain groups over any field F, though the natural analogues of the Ramanujan-Petersson conjecture have been demonstrated to be false. However, the most important cases of the conjecture remain open to this day, even within the classical theory of Maass forms.

Further Reading

We hope this vague, sketchy, and mathematically disconnected survey of the Ramanujan-Petersson conjecture was, to some extent, readable. Many concepts mentioned in the above stretch far beyond the author's current intellectual capacity.

To conclude the survey, we should mention several sources of further reading for the interested reading:

- Daniel Bump [6] has a textbook *Automorphic Forms and Representations*, which seems to be cementing itself as a standard reference for the topic.
- James Arthur [10] has a fairly detailed description of modern progress on the automorphic representation theory of certain matrix groups.
- Luis Lomelí [9] wrote an excellent survey on the interplay between the generalised Ramanujan-Petersson conjectures and the Langlands program.
- Winnie Li [8] has a detailed expository article on the implications of the Ramanujan-Petersson conjecture, with a focus on its applications to surprising areas such as graph theory.

This exposition was perhaps not at the level of detail and mathematical maturity that I wanted it to be at, but reading these sources in particular has shown me many completely new types of mathematics that I hadn't seen before.

References

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