Classical Rigid Analytic Spaces and Rigidifications

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Rigid spaces were first introduced by Tate in the study of the multiplicative uniformisation of elliptic curves over a nonarchimedean field. In the archimedean setting, it is well known that an elliptic curve $E/\mathbb{C}$ is isomorphic to $\mathbb{C}^\times / \langle q \rangle$, where $q = e^{2\pi i z}$. Attempts to generalise this construction to nonarchimedean settings quickly run into significant topological barriers, and this motivated Tate to create the theory of rigid analytic spaces.

Since then, the development and applications of this theory have only expanded. The applications of rigid analytic geometry are too broad and numerous to list in good faith, but one that must be mentioned is its use in the theory of overconvergent modular forms.

Serre's p-adic modular forms suffered from a continuous spectrum of the U_p operator; deeper study necessitates the restriction to some subspace of these modular forms. Katz [4] realised that Serre's p-adic modular forms were those that resembled analytic functions that converge on the space of elliptic curves with good reduction modulo p. This is a subset of the modular curve X over $\mathbb{C}_p$.

Katz's theory of overconvergent modular forms restricts its view to the modular forms that converge on a set slightly larger than the elliptic curves with good reduction; Calegari [2] and other sources call these the "not too supersingular" elliptic curves. These subsets of X are not compatible with the usual Zariski topology; moreover, due to the totally disconnected nonarchimedean topology, local-to-global correspondences involving power series are untenable. For these reasons, one must "rigidify" X and work on the corresponding rigid analytic space in order to make precise these geometric intuitions.

Our goal is thus to give a gentle introduction to Tate's rigid analytic spaces and ultimately state the analogue of GAGA from complex analytic geometry; this constitutes the "classical" theory. For the interested reader, [1] (our primary reference) gives a more or less complete account of this theory. [5] describes Tate's original application and motivations of nonarchimedean multiplicative uniformisations, and [3] is a thorough exposition with detailed examples of rigid analytic spaces in general. Sources such as [6] and [2] do a great job illustrating the importance of rigid analytic geometry to overconvergent modular forms, though their emphasis is placed primarily on the latter topic.

It should be mentioned that Tate's construction was highly explicit. Modern applications favour Raynaud's perspective of rigid analytic geometry, which develops these spaces as generic fibres of certain formal schemes. For brevity, we shall not focus on this alternative perspective; we refer the interested reader to [1].

Affinoid K-algebras and K-spaces

Analogous to how schemes are topological spaces that locally resemble affine varieties, rigid spaces are those that locally resemble affinoids, which we shall define shortly. One should vaguely envision affinoids as analytic analogues to affine varieties — rather than working with rings of polynomials, we shall be working with rings of converging power series. This allows us to make sense of "analytic functions" over nonarchimedean fields while simultaneously avoiding intrinsically local concepts of derivatives and analyticity.

In what follows, K will be a field with a complete nonarchimedean absolute value.

Definition. *The Tate algebra over K, denoted $K\langle X_1, \ldots, X_n\rangle$ or T_n for $n \geq 1$, is the set of all formal power series $\sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha \in K[\![X_1, \ldots, X_n]\!]$ such that $\lim_{|\alpha|\to\infty} |c_\alpha| = 0$. An affinoid K-algebra is any quotient of T_n.*

T_n can be naturally identified with the power series on K^n that converge on the closed unit disc. Many constructs from elementary algebraic geometry indeed carry over, such as the Zariski topology on the spectrum of T_n, a correspondence between ideals of T_n and vanishing sets in the closed unit disc, and identifications between affinoid K-algebras and function rings on Zariski-closed subsets of the unit disc. [1] develops this theory in great detail in chapter 3.

Importantly, T_n is complete with respect to the Gauss norm, given by

$$\left|\sum_{\alpha\in\mathbb{N}^n} c_\alpha X^\alpha\right| := \sup_{\alpha\in\mathbb{N}^n} |c_\alpha|.$$

Its quotients are endowed with the usual quotient norm, and thus all affinoid K-algebras are in fact Banach K-algebras.

An affinoid K-space is just a set of the form $\operatorname{Spm} A$. We shall produce a suitable topology and a sheaf of K-algebras later. A morphism of affinoid K-spaces $\operatorname{Spm} A \to \operatorname{Spm} B$ can be induced by a K-algebra homomorphism $\varphi : B \to A$. This is because if $\mathfrak{m}$ is a maximal ideal of A, then there are natural inclusions $K \hookrightarrow B/\varphi^{-1}(\mathfrak{m}) \hookrightarrow A/\mathfrak{m}$. $A/\mathfrak{m}$ is a finite extension of K, and it can be shown that $B/\varphi^{-1}(\mathfrak{m})$ must be a field. Thus, we define a morphism of affinoid K-spaces to just be the data of the underlying K-algebra homomorphisms.

The Zariski topology is a rather coarse topology to work with, and it is somewhat unnatural in the sense that it bears little resemblance to the norm topology on K^n. This is in spite of the natural identification of $\operatorname{Spm} T_n$ with the closed ball in K^n! There is a different topology that one can place on $\operatorname{Spm} T_n$ that "arises from" the topology on K^n.

Specifically, let $X = \operatorname{Spm} A$. For a maximal ideal $\mathfrak{m} \in X$ and $f \in A$, define $f(\mathfrak{m})$ to be the reduction of f modulo $\mathfrak{m}$. Using the quotient norm on $A/\mathfrak{m}$ (noting that $\mathfrak{m}$ will always be closed in the norm topology), we define the sets

$$X(f, \epsilon) = \{\mathfrak{m} \in X : |f(\mathfrak{m})| \leq \epsilon\}.$$

Definition. *The canonical topology on $X = \operatorname{Spm} A$ is the topology generated by the sets $X(f, \epsilon)$ for $f \in A$ and $\epsilon > 0$. A rational domain is a set of the form*

$$X\left(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}\right) := \{\mathfrak{m} \in X : |f_i(\mathfrak{m})| \leq |f_0(\mathfrak{m})| \ \ \forall i = 1, \ldots, r\},$$

where $f_0, \ldots, f_r \in A$ have no common zeroes.

It can be verified that rational domains are open in the canonical topology. Upon making the natural identifications of $\operatorname{Spm} T_n$ with the closed unit ball in K^n, the canonical topology resembles a weak topology induced by the norm. Indeed, this canonical topology is far finer than the Zariski topology and is totally disconnected (much like the norm topology on K^n).

Affinoid Subdomains and Grothendieck Topologies

Let us now return to the issue of topologies. Where schemes are constructed by gluing together affine varieties (which are themselves locally ringed spaces), we would like to realise rigid spaces as glued-together affinoid spaces with the additional structure of a ringed space.

We'll remark here that the desired to make affinoid spaces into ringed spaces requires that we work with the maximal spectrum instead of $\operatorname{Spec} A$. Localisations at non-maximal prime ideals do not interact well with the norm, and it is desirable for the localisations to remain complete. As such, a "completed localisation" can be defined instead, and although this construction works well with maximal ideals, it no longer permits localisations at non-maximal primes.

On one hand, the canonical topology is far too fine for this to be possible: it is totally disconnected, so sheaves on $\operatorname{Spm} A$ will typically degenerate. Specifically, if $X = \operatorname{Spm} A$, we would like to give X the sheaf of rings $\mathscr{O}_X$, defined just like the classical algebro-geometric construction for affine schemes. However, this is merely a presheaf due to these topological barriers.

On the other hand, the Zariski topology on affinoid spaces is too coarse. When constructing schemes, distinguished open sets of the form $\{f(x) \neq 0\}$ play a central role, for every open set decomposes into a *finite* union of distinguished open sets. The simplicity of these open sets together with this finite-ness condition are what allow one to formalise the gluing process. Unfortunately, affinoid spaces do not admit Zariski-open anologues to these distinguished open sets.

Hence, our aim is to construct a topology finer than the Zariski topology to allow for these well-behaved open sets but simultaneously coarser than the canonical topology to allow for interesting sheaf structures to arise.

Definition. *Let $X = \operatorname{Spm} A$ be an affinoid K-space. $U \subseteq X$ is an affinoid subdomain if there exists some affinoid K-space X' with a morphism $\iota : X' \to X$ with $\iota(X') \subseteq U$ satisfying the following universal property: if $f : Y \to X$ is any morphism of affinoid K-spaces such that $f(Y) \subseteq U$, then f factors into $Y \to X' \xrightarrow{\iota} X$.*

The motivation for this definition is twofold: first, affinoid subdomains are "compatible" with the presheaf $\mathscr{O}_X$. Second, affinoid subdomains admit finite decompositions into "simple" open subsets that are anologous to the distinguished open subsets of an affine scheme. These two facts, together with a technical refinement of the toploogy, will allow for a meaningful notion of a rigid space. To formalise these statements, we shall state two theorems:

Theorem (Tate's Acyclicity Theorem)**.** *Let X be an affinoid K-space, and let $\{U_i\}$ be a finite cover of X by affinoid subdomains. Let $U \subseteq X$ be open in the canonical topology, and let $V_i = U \cap U_i$. If $f_i \in \mathscr{O}_X(V_i)$ such that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$ for all i, j, then there exists a unique $f \in \mathscr{O}_X(U)$ such that $f|_{V_i} = f_i$ for all i.*

Theorem (Gerritzen-Grauert)**.** *Every affinoid subdomain in an affinoid K-space is a finite union of rational domains.*

There is a more general statement of Tate's acyclicity theorem, but this version is sufficient for our purposes. The key takeaway is that uniqueness and gluability are only guaranteed to hold for finite coverings by affinoid subdomains. Hence, one cannot expect to make $\mathscr{O}_X$ into a sheaf on any topology containing all affinoid subdomains without a substantially stronger result. Rather than undertaking this effort, one broadens the notion of a topology to restrict the types of open covers permitted within the topology. This is the Grothendieck topology:

Definition. *A Grothendieck topology $\mathfrak{T}$ is a category* $\operatorname{Cat}\mathfrak{T}$ *and a set* $\operatorname{Cov}\mathfrak{T}$*. Objects of* $\operatorname{Cat}\mathfrak{T}$ *are called admissible opens. Elements of* $\operatorname{Cov}\mathfrak{T}$*, called coverings, are families of morphisms in* $\operatorname{Cat}\mathfrak{T}$ $\{U_i \to U\}_{i\in I}$ *satisfying:*

- *If $U \to V$ is an isomorphism, $\{U \to V\} \in \operatorname{Cov}\mathfrak{T}$.*
- *If $\{U_i \to U\}_{i\in I}$ and and $\{V_{ij} \to U_i\}_{j\in J_i}$ are coverings, then the composite*

$$\{V_{ij} \to U_i \to U\}_{i\in I, j\in J_i}$$

  is also a covering.
- *If $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$, then the fibred products $U_i \times_U V$ all exist and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.*

A set X with a Grothendieck topology $\mathfrak{T}$ where $\operatorname{Cat}\mathfrak{T}$ is a category of subsets of X is called a G-topological space. In this case, we call elements of $\operatorname{Cov}\mathfrak{T}$ admissible coverings to distinguish from arbitrary open covers. In particular, any topological space is a G-topological space with $\operatorname{Cat}\mathfrak{T}$ as the category of open subsets and $\operatorname{Cov}\mathfrak{T}$ defined naturally. A function $f : X \to Y$ between G-topological spaces is continuous if f^{-1} takes admissible opens to admissible opens and admissible covers to admissible covers.

Definition. *Let $(X, \mathfrak{T})$ be a G-topological space. A presheaf on X is a contravariant functor $\mathscr{F}$ on* $\operatorname{Cat}\mathfrak{T}$ *(to a category such as Groups, Rings, etc.). $\mathscr{F}$ is a sheaf if for any covering $\{U_i \to U\}_{i\in I}$ and $f_i \in \mathscr{F}(U_i)$ satisfying $f_i|_{U_i\times_U U_j} = f_j|_{U_i\times_U U_j}$, there is a unique $f \in \mathscr{F}(U)$ satisfying $f|_{U_i} = f_i$.*

Put another way, sheaves on G-topological spaces satisfy the gluability axiom on admissible covers. Naïvely, if X is an affinoid K-space, one may make it a G-topological space by declaring Cat $\mathfrak{T}$ the category of affinoid subdomains with inclusions as morphisms and Cov $\mathfrak{T}$ the set of all finite coverings of affinoid subdomains by affinoid subdomains. Then indeed $\mathscr{O}_X$ is a sheaf on X by Tate acyclicity, but the topology is still too coarse.

Specifically, if $f : X \to Y$ is a map of affinoid K-spaces, then f may not be continuous with respect to the above Grothendieck topologies. The admissible opens and admissible coverings must be relaxed slightly as follows:

Definition. *Let X be an affinoid K-space. Declare $U \subseteq X$ to be an admissible open if it admits a possibly infinite covering by affinoid subdomains $U = \bigcup_{i \in I} U_i$ such that whenever $f : X' \to X$ is a map of affinoid K-spaces with $f(X') \subseteq U$, there is a finite covering of affinoid subdomains $X' = \bigcup V_j$ such that for all i, $f^{-1}(V_i) \subseteq V_j$ for some j.*

A covering $U = \bigcup_{i \in I} U_i$ with each U, U_i admissible is an admissible covering if for each $f : X' \to X$ a map of affinoid K-spaces with $f(X') \subseteq U$, the same conclusion as above holds. The resulting Grothendieck topology $\mathfrak{T}$ is called the strong Grothendieck topology on X.

$\mathscr{O}_X$ is not *a priori* a sheaf on the strong Grothendieck topology, but again one may mimic algebraic geometry and construct a sheafification of any presheaf. Then $\mathscr{O}_X^{\mathrm{sh}}$ (the sheafification) is a *bona fide* sheaf on $\mathfrak{T}$, and moreover, if U is an affinoid subdomain of X, $\mathscr{O}_X(U) = \mathscr{O}_X^{\mathrm{sh}}(U)$. We shall abuse notation and identify $\mathscr{O}_X$ with $\mathscr{O}_X^{\mathrm{sh}}$. This is called the sheaf of rigid analytic functions on X. Other sheaf-theoretic notions (e.g. stalks, locally ringed spaces, morphisms of locally ringed spaces, etc.) all carry over in the natural ways to this setting.

This completes our definition of the rigid-analytic analogue to affine varieties: where the latter is a locally ringed space $(Y, \mathscr{O}_Y)$ that is isomorphic to Spec R with its sheaf of functions, an affinoid K-space is a G-topological space Y with a sheaf of K-algebras $\mathscr{O}_Y$ whose stalks are all local rings that is isomorphic to Spm A with its sheaf of rigid analytic functions.

Rigid Spaces and Ridigification

The definition of a rigid space over K is now quite natural: it is simply a locally ringed G-topological K-space $(X, \mathscr{O}_X)$ (i.e., $\mathscr{O}_X$ is a sheaf of K-algebras whose stalks are all local rings) with an admissible covering $\{X_i \to X\}_{i \in I}$ such that $\left(X_i, \mathscr{O}_X|_{X_i}\right)$ is isomorphic to an affinoid K-space as locally ringed G-topological K-spaces. The Grothendieck topology on X is subject to some mild technical assumptions.

In words, a rigid space is just something that is locally an affinoid K-space. These are generalisations of complex analytic spaces, as the "functions" in $\mathscr{O}_X$ are those that locally resemble converging power series. We should remark that this construction has addressed all of our topological concerns with nonarchimedean geometry, and the odd topological constructions and definitions were all made so that there would be a sensible local-to-global connection between a function and its power series.

To conclude, we shall briefly describe the process of rigidification. We start with a scheme Z locally of finite type over K; we would like to associate to it a rigid analytic space Z^{rig}. Specifically, it is a rigid space with a map of locally ringed G-topological K-spaces $Z^{\mathrm{rig}} \to Z$ such that any map of locally ringed G-topological K-spaces $f : Y \to Z$ factors uniquely through Z^{rig}.

First, one rigidifies affine n-space $\mathbb{A}^n_K$. Intuitively, the ridigification is just K^n with the sheaf of globally converging power series in n variables. To describe this as a rigid space, first fix some $c \in K^n$ with $|c| > 1$ and define B_i to be the ball of radius $|c|^i$. Define $T_n^{(i)} = K\left\langle c_1^{-1}X_1, \ldots, c_n^{-1}X_n\right\rangle$, i.e. the K-algebra of power series that converge on B_i. Then, since $K^n = \bigcup_{i=0}^{\infty} B_i$, one would like to write "$\mathbb{A}_K^{n,\mathrm{rig}} = \bigcup_{i=0}^{\infty} \operatorname{Spm} T_n^{(i)}$". One can then verify that indeed the right-hand-side describes a rigid space, for there are the natural inclusions

$$\operatorname{Spm} T_n^{(0)} \hookrightarrow \operatorname{Spm} T_n^{(1)} \hookrightarrow \cdots$$

This argument extends naturally to when Z is an affine scheme. If $Z = \operatorname{Spec} K\left[X_1, \ldots, X_n\right]/I$ for some ideal I, we let $\mathfrak{I}^{(i)} \subseteq T_n^{(i)}$ be the ideal generated by elements of I. Note there is an obvious and natural inclusion of $K\left[X_1, \ldots, X_n\right] \hookrightarrow T_n^{(i)}$ for all i. Then, one repeats the above process and declares $Z^{\mathrm{rig}} = \bigcup_{i=0}^{\infty} \operatorname{Spm} T_n^{(i)}/\mathfrak{I}^{(i)}$. An analogous chain of inclusions justifies why this union is a rigid space.

By verifying the universal property of rigidifications mentioned earlier, one sees that this construction is independent of the choice of c. Moreover, one can rigidify any K-scheme along the affine open subsets, and the universal property ensures that the rigidifications agree along the intersections.

Concluding Remarks

As a final remark, we should briefly expound upon one of the applications to overconvergent modular forms. In [2], X_r (the space of r-overconvergent modular forms) is defined as a certain subset of X^{rig}, the rigidification of the modular curve X over $\mathbb{Q}_p$. X^{ord} is the ordinary locus of X, whose $\mathbb{Q}_p$-points correspond to elliptic curves with good reduction modulo p. There is a natural identification of $\mathbb{Q}_p$-points of X and X^{rig}, so one can identify $X^{\mathrm{ord}}\left(\mathbb{Q}_p\right)$ with the corresponding points in X^{rig}.

From there, one can use the Hasse invariant to demonstrate that X^{ord} is an *affinoid* open subset of X^{rig}. Its complement is a collection of closed discs containing supersingular elliptic curves. X_r is then defined as an enlargement of X^{ord} by removing smaller and smaller discs from X^{rig}. Of course, the rigidification is necessary here, as this construction is incompatible with the Zariski topology on X.

This concludes our discussion of rigid analytic spaces from the classical perspective of Tate. This is by no means a complete account of the theory; again, we refer the reader to [1] for a thorough development of what was outlined above. Nevertheless, we hope this was able to provide a gentle introduction to the constructions and intuition behind rigid analytic geometry.

References

[1] Siegfried Bosch. *Lectures on Formal and Rigid Geometry*. Springer, 2014.

[2] Frank Calegari. *Congruences Between Modular Forms*. 2013.

[3] Brian Conrad. "Several approaches to non-archimedean geometry". In: (Aug. 2008). DOI: 10.1090/ulect/045/02.

[4] Nicholas Katz. *p-adic Properties of Modular Schemes and Modular Forms*. Springer, Berlin, Heidelberg, 1973.

[5] Mihran Papikian. "Rigid Analytic Geometry and Abelian Varieties". In: 2004.

[6] Jan Vonk. "Overconvergent Modular Forms and Their Explicit Arithmetic". In: *Bulletin of the American Mathematical Society* 58.3 (July 2021), pp. 313 –356.