Exceptional Characters in Sharifi's Conjecture

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Sharifi's conjecture is a refinement of the Iwasawa main conjecture, and it has been the focus of much recent work. Of particular note is Fukaya and Kato's work [2], which made significant progress towards the conjecture. Their work relied heavily on the assumption that a certain Dirichlet character was non-exceptional. Our goal is to give a moderately detailed outline of Shih and Wang's progress in the case where this assumption fails.

Let us very briefly introduce the conjecture and Fukaya and Kato's work in more detail. Throughout, N is a positive integer, $p \geq 5$ is a prime such that $p \nmid N\varphi(N)$ (where φ is the Euler totient function). For each $r > 0$, one can define maps

$$\varpi_r : H_1\left(X_1\left(Np^r\right)_{/\overline{\mathbb{Q}}}, C_r^0, \mathbb{Z}\right)^{+,\mathrm{ord}} \otimes \mathbb{Z}_p \to K_2\left(\mathbb{Z}\left[\zeta_{Np^r}, \frac{1}{Np}\right]\right)^+ \otimes \mathbb{Z}_p$$

by mapping the Manin symbol $[u:v]$ to the Steinberg symbol $\left\{1-\zeta_{Np^r}^u, 1-\zeta_{Np^r}^v\right\}$. Here, $X_1\left(Np^r\right)$ is the closed modular curve, C_r^0 are the cusps away from 0, and ζ_{Np^r} is a fixed primitive Np^r-th root of unity.

The codomain is isomorphic to $H^2_{\text{ét}}\left(\mathbb{Z}\left[\zeta_{Np^r}, \frac{1}{Np}\right], \mathbb{Z}_p(2)\right)$. Identifying the two, the Manin symbol $[u:v]$ is instead sent to the cup product $\left(1-\zeta_{Np^r}^u, 1-\zeta_{Np^r}^v\right)^+$. To ensure that the map is compatible with certain module structures, one needs to twist the domain or codomain. One can "ascend the towers" and take limits in r, yielding the map

$$\varpi : \varprojlim_r H_1\left(X_1\left(Np^r\right)_{/\overline{\mathbb{Q}}}, C_r^0, \mathbb{Z}\right)^{-,\mathrm{ord}}(1) \otimes \mathbb{Z}_p \to \varprojlim_r H^2_{\text{ét}}\left(\mathbb{Z}\left[\zeta_{Np^r}, \frac{1}{Np}\right], \mathbb{Z}_p(2)\right)^+.$$

Following Shih and Wang's notation, we set

$$H = \varprojlim_r H_1\left(X_1\left(Np^r\right)_{/\overline{\mathbb{Q}}}, C_r^0, \mathbb{Z}\right)^{\mathrm{ord}} \otimes \mathbb{Z}_p \qquad \text{and} \qquad S = \varprojlim_r H^2_{\text{ét}}\left(\mathbb{Z}\left[\zeta_{Np^r}, \frac{1}{Np}\right]\right).$$

We let $\mathfrak{h}$ be the (dual) Hecke algebra acting on H, and $I \subseteq \mathfrak{h}$ be the Eisenstein ideal. Let $\Lambda = \mathbb{Z}_p\left[(\mathbb{Z}/Np\mathbb{Z})^\times\right][\![1+p\mathbb{Z}_p]\!]$, and note that $\varpi : H^-(1) \to S$ is a homomorphism of Λ-modules.

Sharifi conjectured that ϖ was Eisenstein (and this was proved by Fukaya and Kato); moreover, he conjectured that it induced an isomorphism of Λ-modules

$$\varpi : H^-(1)/IH^-(1) \xrightarrow{\sim} S.$$

For an even primitive Dirichlet character θ of modulus Np, we define $R_\theta = \mathbb{Z}_p[\operatorname{im}\theta]$ as a $\mathbb{Z}_p\left[(\mathbb{Z}/Np\mathbb{Z})^\times\right]$-module with $(\mathbb{Z}/Np\mathbb{Z})^\times$ acting via θ. For any $\mathbb{Z}_p\left[(\mathbb{Z}/Np\mathbb{Z})^\times\right]$-module M, we define its θ-eigenspace as $M_\theta = M \otimes_{\mathbb{Z}_p\left[(\mathbb{Z}/Np\mathbb{Z})^\times\right]} R_\theta$.

One can show that H^- and S decompose into a direct sum of their θ-eigenspaces as θ ranges over all such characters, and this induces maps

$$\varpi_\theta : H_\theta^-(1)/I_\theta H_\theta^-(1) \to S_\theta.$$

Let ω be the Teichmüller character. We say a primitive character θ of conductor Np is "exceptional" if $\theta = \omega$ when restricted to $(\mathbb{Z}/p\mathbb{Z})^\times$ and $\theta(p) = \omega(p)$ when restricted to $(\mathbb{Z}/N\mathbb{Z})^\times$; we say θ is nonexceptional otherwise.

Fukaya and Kato made significant progress when θ is nonexceptional. Sharifi had constructed a map $\Upsilon_\theta : S_\theta \to H_\theta^-(1)/I_\theta H_\theta^-(1)$ which he conjectured to be the inverse to ϖ_θ. Fukaya and Kato showed that

$$\xi_\theta' \Upsilon_\theta \circ \varpi_\theta = \xi_\theta'$$

modulo p-torsion, where ξ_θ is a certain p-adic L-function. However, it is the construction of Υ_θ that relies heavily on θ being nonexceptional, as it requires Ohta's work to understand the relationship between Galois and Hecke actions on certain Λ_θ-modules. These technical barriers prevent this work from being directly extended to the case when θ is exceptional.

We will now assume that θ is exceptional, and we write $\chi = \theta\omega^{-1}$.

Localisations of $H_\theta^-(1)$ and S_θ

Let γ be a topological generator of $\mathrm{Gal}\left(\mathbb{Q}\left(\zeta_{Np^\infty}\right)/\mathbb{Q}\left(\zeta_{Np}\right)\right)$ and $\kappa : \mathrm{Gal}\left(\mathbb{Q}\left(\zeta_{Np^\infty}\right)/\mathbb{Q}\left(\zeta_N\right)\right) \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character. One has $\Lambda_\theta \cong \mathbb{Z}_p[\theta][\![T]\!]$ via $\kappa(\gamma) \mapsto T+1$; we will now identify Λ_θ with $\mathbb{Z}_p[\theta][\![T]\!]$. Let $\mathfrak{p}$ be the height one prime ideal generated by $T+1-\kappa(\gamma)$.

Let $\Lambda_{\theta,\mathfrak{p}}$ be the localisation at $\mathfrak{p}$, and let $k_{\theta,\mathfrak{p}}$ be its residue field. One of the main results of Shih and Wang, and the focus of our discussion, is as follows:

Theorem.

$$\varpi_{\theta,\mathfrak{p}} : H_{\theta,\mathfrak{p}}^-(1)/I_{\theta,\mathfrak{p}} H_{\theta,\mathfrak{p}}^-(1) \to S_{\theta,\mathfrak{p}}$$

is an isomorphism of $k_{\theta,\mathfrak{p}}$-vector spaces.

Rather than constructing some $\Upsilon_{\theta,\mathfrak{p}}$, Shih and Wang first demonstrate that both the domain and codomain are 1-dimensional over $k_{\theta,\mathfrak{p}}$. Then, they constuct an element of H_θ^- whose image under ϖ_θ remains nontrivial after localising. We will give a brief overview of the localisations $H_{\theta,\mathfrak{p}}^-$ and $S_{\theta,\mathfrak{p}}$, beginning with the latter.

Let $K = \mathbb{Q}\left(\zeta_{Np^\infty}\right)$, and let Σ be the (finite) set of places of K above p. Let $X_{K,\Sigma}$ be the Galois group of the maximal abelian pro-p extension of K where every prime above those in Σ split completely. There is an exact sequence

$$0 \to X_{K,\Sigma} \to \varprojlim_r H^2\left(\mathbb{Z}\left[\zeta_{Np^r}, \frac{1}{p}\right], \mathbb{Z}_p(1)\right) \to \bigoplus_\Sigma \mathbb{Z}_p \to \mathbb{Z}_p \to 0.$$

After restricting to the χ-component and applying a Tate twist, the second nonzero term becomes S_θ. Writing

$$\bigoplus_{\Sigma}^{0} \mathbb{Z}_p = \ker\left(\bigoplus_{\Sigma} \mathbb{Z}_p \to \mathbb{Z}_p\right),$$

they arrive at the short exact sequence

$$0 \to X_{K,\Sigma,\chi}(1) \to S_\theta \to \left(\bigoplus_{\Sigma}^{0} \mathbb{Z}_p\right)_\chi (1) \to 0.$$

On one hand, by analysing the characteristic ideal of $X_{K,\chi}/X_{K,\Sigma,\chi}$, one can show that $X_{K,\Sigma,\chi}(1)_{\mathfrak{p}} = 0$. On the other hand, one can show that

$$\left(\bigoplus_{\Sigma}^{0} \mathbb{Z}_p\right)_\chi (1) \cong \Lambda_\theta/\mathfrak{p}$$

as Λ_θ-modules by analysing the Galois action; localising and returning to the previous exact sequence yields $S_{\theta,\mathfrak{p}} \cong k_{\theta,\mathfrak{p}}$ as $k_{\theta,\mathfrak{p}}$-vector spaces.

Showing the domain is 1-dimensional is more involved. We begin by defining $G_{\theta^{-1}} \in \Lambda_\theta$ to be the power series satisfying $G_{\theta^{-1}}\left(\kappa(\gamma)^s - 1\right) = L_p\left(-s-1, \theta^{-1}\omega^2\right)$, where L_p is the Kubota-Leopoldt p-adic L-function. We then define $\xi_\theta(T) = G_{\theta^{-1}}\left((T+1)^{-1} - 1\right) \in \Lambda_\theta$. Following the work of Ohta and Lafferty, Shih and Wang were able to demonstrate that $\mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta/\left(\xi_\theta\right)$.

Due to the exceptionality of θ, it is known that ξ_θ has a simple zero at $T = \kappa(\gamma) - 1$; this is called the trivial zero. Recalling that $\mathfrak{p}$ was defined as the ideal generated by $T + 1 - \gamma(\kappa)$ and using the fact that this zero is simple, we see that localising $\Lambda_\theta/\left(\xi_\theta\right)$ at $\mathfrak{p}$ gives a field, hence by the previous isomorphism, $I_{\theta,\mathfrak{p}}$ is the maximal ideal of $\mathfrak{h}_{\theta,\mathfrak{p}}$. In fact, Shih and Wang make the stronger claim (citing [1]'s theorem 3.5(ii)) that $\mathfrak{h}_{\theta,\mathfrak{p}} \cong \Lambda_{\theta,\mathfrak{p}}$ and $I_{\theta,\mathfrak{p}} \cong \mathfrak{p}$.

Finally, using the structure of H_θ as a free rank-2 $\mathfrak{h}_\theta$-module and the Gorenstein property of $\mathfrak{h}_{\theta,\mathfrak{p}}$, they conclude that both $H^{\pm}_{\theta,\mathfrak{p}}$ are free $\mathfrak{h}_{\theta,\mathfrak{p}}$-modules. Using the isomorphism $\mathfrak{h}_{\theta,\mathfrak{p}} \cong \Lambda_{\theta,\mathfrak{p}}$ and $I_{\theta,\mathfrak{p}} \cong \mathfrak{p}$ again, it follows that $H^-_{\theta,\mathfrak{p}}/I_{\theta,\mathfrak{p}}H^-_{\theta,\mathfrak{p}}$ is a one-dimensional $k_{\theta,\mathfrak{p}}$-vector space.

We will remark that the isomorphism $\mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta/\left(\xi_\theta\right)$ was also stated in the nonexceptional case covered in lecture. However, Shih and Wang's choice of ξ_θ is modified and takes advantage of the trivial zero of the Kubota-Leopoldt L-function. In particular, Shih and Wang take $\xi_\theta = L_p\left(-2, \theta^{-1}\omega^2\right)$ as opposed to $L_p\left(-1, \theta^{-1}\omega^{-2}\right)$. The rationale and motivation behind this difference is unfortunately unclear to the author.

The Image of a Modular Symbol Under $\varpi_{\theta,\mathfrak{p}}$

In the previous section, we described how

$$\varpi_{\theta,\mathfrak{p}} : H^-_{\theta,\mathfrak{p}}(1)/I_{\theta,\mathfrak{p}}H^-_{\theta,\mathfrak{p}}(1) \to S_{\theta,\mathfrak{p}}$$

is a map between two one-dimensional $k_{\theta,\mathfrak{p}}$-vector spaces. Hence, it is enough to show that $\varpi_{\theta,\mathfrak{p}}$ has nontrivial image to deduce that it is an isomorphism.

To do so, Shih and Wang consider the short exact sequence of $\Lambda_\theta\left[G_{\mathbb{Q}}\right]$-modules (where $G_{\mathbb{Q}} = \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$)

$$0 \to H_\theta \to \varprojlim_r H^1_{\text{ét}}\left(Y_1\left(Np^r\right)_{/\overline{\mathbb{Q}}}, \mathbb{Z}_p\right)^{\text{ord}}_\theta \to e\cdot \mathbb{Z}_p\left[\theta\right]\llbracket C_\infty\rrbracket\left(-1\right) \to 0,$$

where $Y_1\left(Np^r\right)$ is the usual modular curve, e is Hida's idempotent, and

$$\mathbb{Z}_p\left[\theta\right]\llbracket C_\infty\rrbracket = \varprojlim_r \mathbb{Z}_p\left[\theta\right]\left[C_1\left(Np^r\right)\right].$$

The maps in the limits are the natural projections to cusps of lower level. They then consider a certain rank 1 Λ_θ-submodule of $e\cdot\mathbb{Z}_p\left[\theta\right]\llbracket C_\infty\rrbracket$, denoted $\Lambda_\theta e_{(\theta,1)}$ and determined by the residue of a certain Eisenstein series, and they set $\widetilde{H}_\theta$ to be the preimage of $\Lambda_\theta e_{(\theta,1)}$ in $\varprojlim_r H^1_{\text{ét}}\left(Y_1\left(Np^r\right)_{/\overline{\mathbb{Q}}}, \mathbb{Z}_p\right)^{\text{ord}}_\theta$. The prior short exact sequence can then be refined to

$$0 \to H_\theta \to \widetilde{H}_\theta \to \Lambda_\theta \cdot e_{(\theta,1)}(-1) \to 0.$$

We remark here that analysing this exact sequence and several related sequences underpins the Shih and Wang's proof that $\mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta/\left(\xi_\theta\right)$, but for brevity and presentation we omitted these details.

Let $\mathfrak{H}_\theta$ be the Hecke algebra acting on $\widetilde{H}_\theta$, and let $\mathfrak{I}_\theta$ be the Eisenstein ideal. By tensoring the above short exact sequence $-\otimes_{\mathfrak{H}_\theta}\mathfrak{h}_\theta$ (known as the Drinfeld-Manin modification), together with the fact that $\mathfrak{H}_\theta/\mathfrak{I}_\theta \cong \Lambda_\theta\cdot e_{(\theta,1)}$, one obtains

$$0 \to H_\theta \to \widetilde{H}_\theta \otimes_{\mathfrak{H}_\theta} \mathfrak{h}_\theta \to \mathfrak{h}_\theta/I_\theta \to 0.$$

In particular, using $\mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta/\left(\xi_\theta\right)$, one sees that

$$\left(\widetilde{H}_\theta \otimes_{\mathfrak{H}_\theta} \mathfrak{h}_\theta\right)/H_\theta \cong \Lambda_\theta/\left(\xi_\theta\right).$$

Most importantly, one obtains as a consequence that $\xi_\theta\left\{0,\infty\right\}_\theta \in H_\theta$ and $\left(1-U_p\right)\cdot\left\{0,\infty\right\}_\theta \in H_\theta$, where $\left\{0,\infty\right\}_\theta$ is the usual modular symbol and $q \mid Np$. To be more precise, one must take the Drinfeld-Manin modification of $\left\{0,\infty\right\}_\theta$, but we will supress this for notational clarity.

It can then be computed that

$$\varpi_\theta\left(\left(1-U_q\right)\cdot\left\{0,\infty\right\}_\theta\right) = \left(q, 1-\zeta_{Np^r}\right)_{r\geq 1,\theta}.$$

The case $q = p$ was proven by Fukaya and Kato in [2], and the remaining cases are a brief computation (see [5]'s corollary 3.4).

Most importantly, this image stays nonzero in $S_{\theta,\mathfrak{p}}$ after localising. To be more specific, Shih and Wang perform an extremely lengthy computation involving Coleman power series, the explicit reciprocity law, and identities involving roots of unity to obtain the formula

$$\left(l, 1-\zeta_N^{p^{-r}}\zeta_{p^r}\right)_{\theta,\mathfrak{p}} = \frac{(p-1)\log_p(l)}{p\varphi(N)}\tau\left(\chi^{-1}\right)L\left(0,\chi\right),$$

where τ is the Gauss sum, $L\left(0,\chi\right)$ is the Dirichlet L-function, $r \geq 0$ and $l \mid N$ is a prime. This is viewed as an element of $\left(\mathcal{O}/p^r\mathcal{O}\right)(1)$, where $\mathcal{O}$ is defined as any extension of $\mathbb{Z}_p\left[\theta\right]$ containing all p-th power roots of unity. The author does not claim to understand some of these computations, and we will treat the high-powered machinery behind it as a black box.

It should be noted that the assumption $\chi(p) = 1$ is essential in this computation. In brief, Shih and Wang determined the projection of $\left(l, 1-\zeta_N^{p^{-r}}\zeta_{p^r}\right)$ by first averaging over the action of $(\mathbb{Z}/Np\mathbb{Z})^\times$, using $\chi(p) = 1$ to decouple the sum over $(\mathbb{Z}/N\mathbb{Z})^\times$ and $(\mathbb{Z}/p\mathbb{Z})^\times$. Later, they use $\chi(p) = 1$ again when reducing a sum into a Dirichlet L-function.

Let q_r be the multiplicative inverse of p^r modulo N. Then,

$$\begin{aligned}
\left(l, 1-\zeta_{Np^r}\right)_{\theta,\mathfrak{p}} &= \left(l, 1-\left(\zeta_N^{p^{-r}}\zeta_{p^r}\right)^{\frac{1}{q_r\cdot p^r+N}}\right) \\
&= \theta\left(q_r\cdot p^r+N\right)^{-1}\left(l, 1-\zeta_N^{p^{-r}}\zeta_{p^r}\right) \\
&= \omega(N)\cdot\frac{(p-1)\log_p(l)}{p\varphi(N)}\tau\left(\chi^{-1}\right)L\left(0,\chi\right).
\end{aligned}$$

Taking r sufficiently large, this becomes nonzero in $\left(\mathcal{O}/p^r\mathcal{O}\right)(1)$, hence also in S_θ. Shih and Wang obtain a similar formula for $\left(p, 1-\zeta_{Np^r}\right)_{\theta,\mathfrak{p}}$, which also remains nonzero for r large.

In particular, we see that $\varpi_{\theta,\mathfrak{p}}\left(\left(1-U_q\right)\{0,\infty\}_\theta\right) \neq 0$ for all $q \mid Np$. Thus $\varpi_{\theta,\mathfrak{p}}$ has nontrivial image, and it follows from the previous section that $\varpi_{\theta,\mathfrak{p}}$ is an isomorphism.

Concluding Remarks

The primary difference with their argument and previous work (e.g., the work of Fukaya and Kato or Sharifi and Venkatesh) is that they could not construct a candidate inverse map Υ_θ. Rather, they were able to exploit the fact that the values of $\varpi_{\theta,\mathfrak{p}}$ could be computed and examined. In contrast, it appears that the values of ϖ_θ when θ is nonexceptional are difficult to obtain; Shih and Wang extensively used $\chi(p) = 1$ and $p \nmid \varphi(N)$ throughout their computations.

It should be stated that (according to Shih and Wang) for height one primes $\mathfrak{q}$ distinct from $\mathfrak{p}$, the arguments of Wake and Wang-Erickson [6] can be used to show that $\varpi_{\theta,\mathfrak{q}}$ is also an isomorphism. However, this is far beyond the scope of the author's understanding, and it is not clear to the author if this together with Shih and Wang's result is enough to imply that ϖ_θ is an isomorphism when θ is not exceptional. Is there a local-to-global argument that can be made, or do primes of height greater than 1 also need to be considered?

In addition, this argument is contingent upon the primitivity of θ and hence χ. In the formula the authors arrive at for the values of $\varpi_{\theta,\mathfrak{p}}$, the factor of $L(0,\chi)$ is known to be nonzero specifically when χ is primitive. Without this assumption, it could vanish.

This concludes our overview of Shih and Wang's theorem. Although we had to omit many, many details, we hope that this gave a reasonable account of Shih and Wang's work and arguments.

References

[1] Adel Betina, Mladen Dimitrov, and Alice Pozzi. *On the failure of Gorensteinness at weight 1 Eisenstein pointns of the eigencurve*. 2021.

[2] Takako Fukaya and Kazuya Kato. *On Conjectures of Sharifi*. 2012.

[3] Takako Fukaya, Kazuya Kato, and Romyar Sharifi. *Modular symbols in Iwasawa theory*. 2015.

[4] Romyar Sharifi. "A reciprocity map and the two-variable p-adic L-function". In: *Annals of Mathematics* 174 (2011), pp. 251 –300.

[5] Sheng-Chi Shih and Jun Wang. *On Sharifi's Conjecture: Exceptional Case*. 2021.

[6] Peterson Wake and Carl Wang-Erickson. "Pseudo-modularity and Iwasawa theory". In: *American Journal of Mathematics* 140.4 (2018), pp. 977 –1040.