Doubly Periodic Meromorphic Functions and the Weierstrass Elliptic Function

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Notation

$\mathbb{C}$	The complex numbers
$\mathbb{C} \cup \{\infty\}$	The Riemann sphere
$\mathbb{R}$	The real numbers
$\mathbb{Z}$	The integers
Λ	A nontrivial complex lattice
D	A fundamental domain of a lattice
∂D	The boundary of a fundamental domain of a lattice
a, b, c	Constants
f, g	Meromorphic functions
j, k, l, m, n	Integers
w, z, z'	Complex Variables
ω_1, ω_2	Generators of a lattice
ω	Any element in a given lattice
$\wp$	The Weierstrass elliptic function
$\mathrm{Ord}_z(f)$	The order of a meromorphic function f at a point z
$\mathrm{Res}_z(f)$	The residue of a meromorphic function f at a point z

1 Introduction and Motivation

During the 17th and 18th centuries, mathematicians such as Euler and Fagnano were interested in the problem of finding the perimeter of an ellipse and arc length of a lemniscate [3]. Physicists were also interested in solving mechanical problems involving pendulums and spinning tops [1]. In both cases, solutions could not be simplified beyond integral expressions of the form $\int R\left(x, \sqrt{P(x)}\right) \mathrm{d}x$, where R is a rational function and P is a third- or fourth-degree polynomial. These are called elliptic integrals, and their inverses are known as elliptic functions [8].

These integrals were reduced to three standard forms by Legendre in the 19th century, and the elliptic functions associated with these standard forms are known as Jacobi's elliptic functions. These are fundamental to elliptic functions in the sense that every elliptic function is related to Jacobi's elliptic functions; they are essentially standard forms of elliptic functions [8]. Amazingly, Abel discovered that all elliptic functions are in fact doubly periodic in the complex plane, allowing for the analysis of these functions through a different perspective and set of techniques [9].

Rather than study elliptic functions through the lens of elliptic integrals, we shall apply the theory of complex analysis to doubly periodic meromorphic functions. Instead of Jacobi's elliptic functions, we will motivate and study the Weierstrass elliptic function, or $\wp$. We will then arrive at the peculiar result that all elliptic functions are rationally expressible in terms of $\wp$ and its derivative. This function is not only fundamental to elliptic functions, but it also has wide applications from physics to crytography.

2 Liouville's Theorems

Definition 1. *A meromorphic function* $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *is* elliptic *or* doubly periodic *if there exist some* $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$, *and* $f(z) = f(z + \omega)$ *for all* $z \in \mathbb{C}$ *and* $\omega \in \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$.

A singly periodic function can be characterised by its period — for instance, we say sin is 2π periodic whereas the complex exponential is $2\pi i$ periodic. This is unique up to multiplication by -1, and so this is a natural way to describe singly periodic functions. However, the two periods of an elliptic function aren't necessarily unique, and this is apparent when we realise that $\omega_1\mathbb{Z} + \omega_2\mathbb{Z} = (\omega_1 + 3817\omega_2)\mathbb{Z} + \omega_2\mathbb{Z}$. We instead characterise these functions by lattices:

Definition 2. $\Lambda \subseteq \mathbb{C}$ *is a* lattice *if there exist some* $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ *such that* $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ *and* $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. *We say* f *is* doubly periodic with respect to Λ *or* Λ-periodic *if* f *is meromorphic on* $\mathbb{C}$ *and* $f(z) = f(z + \omega)$ *for all* $z \in \mathbb{C}$ *and* $\omega \in \Lambda$.

A region of interest is the domain "swept out" by ω_1 and ω_2, for the behaviour of f on one such domain fixes its behaviour everywhere on $\mathbb{C}$.

Definition 3. *Let* Λ *be a lattice, and suppose* $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ *for some* $\omega_1, \omega_2 \in \mathbb{C}$. *Then,* $D = \left\{a\omega_1 + b\omega_2 : a, b \in [0, 1)\right\}$ *is called a* fundamental domain *of* Λ.

Much as the domain of cos can be treated as the quotient $\mathbb{R}/2\pi\mathbb{Z}$, we may treat the domain of our elliptic function as $\mathbb{C}/\Lambda$. The fundamental domain then serves as a set of representatives for this quotient, for no two elements in D will be in the same coset. Notably, we do not constrain an elliptic function f to be holomorphic and instead allow it to be meromorphic — holomorphic functions are too rigid for interesting behaviour under the constraint of double periodicity.

Proposition 4 (Liouville's First Theorem)**.** *Every holomorphic elliptic function is constant.*

Proof. Let Λ be a lattice and let D be a fundamental domain of that lattice. Let f is a holomorphic elliptic function with respect to Λ. Since D is bounded, f is bounded on D, and thus f is bounded on $\mathbb{C}$. By the other Liouville's theorem, f is constant. $\square$

Although meromorphic functions are a much broader class of functions, we still get a fair amount of regularity under the additional constraint of double periodicity. First and foremost, f can only have finitely many poles and zeroes in D (if f is not identically 0, of course). This is because any infinite sequence of zeroes in D will accumulate; a similar argument may be applied to $\frac{1}{f}$ to show the finitude of poles in D. This can be strengthened further yet by the residue theorem:

Theorem 5 (Liouville's Second Theorem)**.** *If Λ is a lattice, D is a fundamental domain of Λ, and f is a meromorphic Λ-periodic function, then $\sum_{z\in D} \operatorname{Res}_z(f) = 0$.*

Proof. If ∂D is the boundary of D and avoids the poles of f,

$$\frac{1}{2\pi i}\int_{\partial D} f(z)\,\mathrm{d}z = \sum_{z\in D} \operatorname{Res}_z(f)$$

Since f can only have finitely many poles, we may perturb ∂D to avoid any poles, if needed. This perturbation should be symmetric in order for us to utilise the double periodicity of f. Due to the double periodicity of f, the integral along the possibly perturbed ∂D vanishes, and we get the result. □

A simple result of this is that no elliptic function can have exactly one simple pole in D. Furthermore, applying this result to the elliptic function $\frac{f'}{f}$ gives:

Corollary 6. *If f is elliptic and not identically zero, the number of zeroes is equal to the number of poles (counting multiplicity); i.e. $\sum_{z\in D} \operatorname{Ord}_z(f) = 0$.*

This can also be proved by using the argument principle. Another more significant result is the last of Liouville's theorems:

Theorem 7 (Liouville's Third Theorem)**.** *f attains every complex value the same number of times in its fundamental domain, counting multiplicities.*

Proof. Let n be the number of zeroes of f. Let $c \in \mathbb{C}$. $f - c$ still has n poles, and so by Liouville's Second Theorem, it must also have n zeroes. □

These theorems impose a great deal of regularity on the behaviour of doubly holomorphic functions on their fundamental domains. When showing equality of two elliptic functions, it is often easier to show that their difference has no poles than directly showing equality, and similar techniques will be employed often in the later sections.

3 The Weierstrass $\wp$ Function

While Liouville's theorems are powerful descriptions of elliptic functions, we still have not seen a non-constant elliptic function. Naturally, we would like to construct some.

We can do this very naïvely by taking any arbitrary meromorphic (but not necessarily elliptic) function f and any lattice Λ and writing

$$g(z) = \sum_{\omega \in \Lambda} f(z + \omega)$$

This sum is doubly periodic with respect to Λ if we may rearrange the sum, and this is only possible if the sum converges absolutely. To help us determine when such a sum converges, we will state a simple convergence test as in [5]:

Lemma 8. *Let $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ be some lattice and let $\varphi(z, \omega)$ be meromorphic in z for any fixed $\omega \in \Lambda$. Suppose further that for any fixed z,*

$$\left|\varphi(z, m\omega_1 + n\omega_2)\right| = O\left(\left(m^2 + n^2\right)^{-c}\right)$$

for some real $c > 1$. Then, the sum

$$\sum_{\substack{m,n \in \mathbb{Z} \\ \varphi(z, m\omega_1 + n\omega_2) \neq \infty}} \varphi(z, m\omega_1 + n\omega_2)$$

converges absolutely and uniformly on compact sets.

The proof is lengthy and computational, but we shall provide a brief outline. Only the terms when m and n are large are considered. The absolute series is rearranged into subseries, each of which is bounded by a converging integral [5]. The case when $c = 1$ may result in series that do not absolutely converge, for the sum varies over $(m, n) \in \mathbb{Z}^2$.

Thus, taking $g(z) = c \sum_{\omega \in \Lambda} (z - \omega)^{-k}$ for any integer $k > 2$ and constant $c \in \mathbb{C}$ gives us a family of meromorphic nonconstant elliptic functions with poles of order k at each point of Λ.

Liouville's second theorem (Theorem 5) shows that every nonconstant elliptic function must have at least two poles counting multiplicity; we would like to construct an elliptic function with exactly one double pole in its fundamental domain. Unfortunately, the naïve choice of $\sum_{\omega \in \Lambda} (z - \omega)^{-2}$ does not converge uniformly, and we cannot apply Lemma 8 because $(z - m\omega_1 - n\omega_2)^{-2} = O\left(m^2 + n^2\right)$.

We may address this by introducing a "corrective" term to ensure convergence, and what we get is the Weierstrass elliptic function, denoted $\wp$.

Definition 9. *The Weierstrass elliptic function associated with a lattice Λ is defined as:*

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}\right)$$

Often, the lattice in question is implied, unambiguous, or redundant, and so it is omitted from the subscript. Clearly, $\wp$ has a double pole on every point in Λ, and if $\wp$ converges uniformly on compact sets, it will be meromorphic on $\mathbb{C}$. Of course, we will also need to verify that $\wp$ is doubly periodic with respect to Λ as well.

Proposition 10. *$\wp$ converges absolutely on $\mathbb{C} \setminus \Lambda$.*

Proof. Fix $z \in \mathbb{C} \setminus \Lambda$, where $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. We may rewrite the summand

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{\omega^2 - (z-\omega)^2}{\omega^2 (z-\omega)^2} = \frac{2\omega z - z^2}{\omega^2 (z-\omega)^2}$$

For any fixed z, we have $\left|2\omega z - z^2\right| = O\left(|\omega|\right)$ while $\left|\omega^2 (z-\omega)^2\right|^{-1} = O\left(|\omega|^{-4}\right)$ whenever $|\omega| >> |z|$. Thus, if we express $\omega = m\omega_1 + n\omega_2$, the summand is

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = O\left(|\omega|^{-3}\right) = O\left(\left(m^2+n^2\right)^{-3}\right)$$

By Lemma 8, this converges uniformly on compact sets. □

Although it is tempting to use a rearrangement argument to show double periodicity, not every term in the summand depends on z; we will instead adapt the argument used in [7] to verify this property.

Proposition 11. *$\wp$ is doubly periodic with respect to Λ.*

Proof. Let $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. We will show that $\wp$ is ω_1-periodic, and by symmetry, any arguments can also be applied to show that it is ω_2-periodic as well. If $z \in \Lambda$, then $\wp(z)$ and $\wp(z+\omega_1)$ both have double poles. Suppose $z \in \mathbb{C} \setminus \Lambda$. Then, we have:

$$\begin{aligned}\wp(z) - \wp(z+\omega_1) =& \frac{1}{z^2} + \sum_{\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) \\ & - \frac{1}{(z+\omega_1)^2} - \sum_{\Lambda\setminus\{0\}} \left(\frac{1}{(z+\omega_1-\omega)^2} - \frac{1}{\omega^2}\right)\end{aligned}$$

The $\frac{1}{\omega^2}$ terms cancel out from both sums, and we may combine terms into the summation to get:

$$= \sum_{\omega\in\Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{(z+\omega_1-\omega)^2}\right)$$

We may apply a similar argument as in the previous proposition to show that this sum is absolutely convergent. Then, by Fubini's theorem, we may rewrite this as an iterated sum and express $\omega = m\omega_1 + n\omega_2$:

$$= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left(\frac{1}{(z - m\omega_1 - n\omega_2)} - \frac{1}{\left(z - (m+1)\,\omega_1 - n\omega_2\right)} \right)$$

The inner sum telescopes, and so we get $\wp(z) - \wp(z + \omega_1) = 0$. Reapplying this argument to ω_2 gives us the desired result. $\square$

It should be fairly clear that $\wp$ is even; this can be shown by using the symmetry of Λ and rearranging the summation (specifically, we can replace ω with $-\omega$). Notably, we may rearrange the series because it is absolutely convergent.

Absolute convergence lets us differentiate the series term-by-term:

Proposition 12. $\wp'(z) = \sum_{\omega \in \Lambda} \frac{-2}{(z-\omega)^3}$

By using the same rearrangement arguments as for $\wp$, one may show that $\wp'$ is odd, and this is vaguely reminiscent of how cos and its derivative $-\sin$ are even and odd. Much like how $\wp$ has double poles precisely at every point in Λ, $\wp'$ has a triple pole precisely at every point in Λ.

Finally, as $\wp$ is meromorphic, it's helpful to see the Laurent series expansion. It is not easy to use the integral formulae to find the series coefficients of $\wp$, and we will instead directly manipulate the expression we have for $\wp$ to resemble a Laurent series. Before that, however, we must establish the Eisenstein series.

Definition 13. *Let $k > 1$ be an integer. Then, the Eisenstein series for a lattice Λ are given by*

$$G_k(\Lambda) = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-2k}$$

We may quickly verify that these series converge by applying Lemma 8. We'll remark that it's common to define $G_k(z) = G_k(z\mathbb{Z} + \mathbb{Z})$ for $z \in \mathbb{H}$, the open upper half-plane. We can quickly apply Lemma 8 again to show that this converges absolutely on compact sets and is thus meromorphic on $\mathbb{H}$. More importantly, these are typical examples of *modular forms*, which are functions on $\mathbb{H}$ that have particular algebraic structures [7]. However, this is quite beyond the scope of our discussion here.

Proposition 14. *The Laurent series expansion of $\wp$ about the origin is given as:*

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\, G_{k+1}(\Lambda)\, z^{2k}$$

Proof. We'll start by rewriting $\wp$ in terms of power series. We may constrain z to a neighbourhood of the origin where $|\omega| > |z|$ for all $\omega \in \Lambda \setminus \{0\}$. As such, we get:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \\
&= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{\omega^2\left(1-\frac{z}{\omega}\right)^2} - \frac{1}{\omega^2} \right)
\end{aligned}$$

Recognising the geometric series allows us to then write

$$\begin{aligned}
&= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{\omega^2} \left(\sum_{k=0}^{\infty} \left(\frac{z}{\omega}\right)^k \right)^2 - \frac{1}{\omega^2} \right) \\
&= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{\omega^2} \sum_{k=1}^{\infty} (k+1) \left(\frac{z}{\omega}\right)^k \right)
\end{aligned}$$

Absolute convergence allows us to switch the order of summation, by Fubini's theorem:

$$= \frac{1}{z^2} + \sum_{k=1}^{\infty} \left((k+1)\, z^k \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-k-2} \right)$$

When k is odd, the symmetry of Λ makes the inner sum vanish; when k is even, we recognise the inner sum to be the Eienstein series. Thus, we get:

$$= \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\, G_{k+1}(\Lambda)\, z^{2k}$$

□

Finally, we shall state without proof the addition formula for the Weierstrass $\wp$ function:

Theorem 15 (Addition Formula)**.** *If $z, z' \in \mathbb{C}$, we have*

$$\wp(z+z') = \frac{1}{4} \left(\frac{\wp'(z) - \wp'(z')}{\wp(z) - \wp(z')} \right)^2 - \wp(z) - \wp(z')$$

The proof has been omitted for brevity's sake. For the avid reader, this result can be proven by finding the Laurent expansion of the difference of either side and observing that there can be at most one pole on the fundamental domain, counting multiplicities. By a corollary of Theorem 5, since the difference is doubly periodic in z, it cannot have any poles. Thus, it must be constant [5].

Thinking back to the motivating topic of elliptic integrals and elliptic functions, the relationship between that and the ideas of doubly periodic meromorphic functions we have been discussing seems tenuous, at best. That being said, the addition formula demonstrates that $\wp$ has some inherent symmetry and structure beyond its double periodicity, and it mimics Euler's finding of addition formulae for elliptic functions [9].

4 Fields of Elliptic Functions

The set of elliptic functions with respect to a fixed lattice Λ form a field, and the field axioms can be quickly verified. Naturally, we're interested in if this field is finitely generated, and if so, what the generators of this field look like. It turns out that this field is generated by only two functions: $\wp$ and $\wp'$.

The method of proof relies on the properties of $\wp$ and $\wp'$, as well as the rigidity of elliptic functions from Liouville's theorems. In brief, we will loosely follow the techniques in [5] and reduce to the case where f is even, then use rational combinations of $\wp$ to "factor out" all of the poles and zeroes of f. We must be careful of when f has a pole or zero on $\frac{1}{2}\Lambda$, for the Λ-periodicity forces each of these to have even order when f is even.

Lemma 16. *If f is an even elliptic function with respect to Λ that is not identically 0, and if $z \in \mathbb{C}$ such that $2z \in \Lambda$, then $\mathrm{Ord}_z(f)$ is even.*

Proof. First, we claim that if f is even, f' is odd. This is because $f'(z) = \frac{\mathrm{d}}{\mathrm{d}z} f(-z) = -f'(-z)$, just by the chain rule. Similarly, if f' is odd, then f'' is even, by a similar argument. More generally, we have that $f^{(n)}(z) = (-1)^n f^{(n)}(-z)$. Each of these derivatives is still Λ-periodic.

Suppose $2z \in \Lambda$, and let k be any odd integer. Then, by Λ-periodicity, we have $f^{(k)}(z) = -f^{(k)}(-z) = -f^{(k)}(-z+2z) = -f^{(k)}(z)$. Thus, $f^{(k)}(z) = 0$ whenever k is odd and $2z \in \Lambda$, and it follows that f must have even order at z. $\square$

Theorem 17. *Every elliptic function with respect to some lattice Λ may be expressed as a rational function of $\wp$ and $\wp'$.*

Proof. Let f be any elliptic function with respect to Λ. We may split f into its even and odd components:

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}$$

Since $\wp'(z)\left(\frac{f(z)-f(-z)}{2}\right)$ is even, and since each term here is clearly elliptic, it suffices to show that every even elliptic function is expressible as a rational function of $\wp$.

Now suppose without loss of generality that f is even. Suppose $z_0 \in \mathbb{C} \setminus \Lambda$ is a zero of f. Let $m = \mathrm{Ord}_{z_0}(f)$. If $2z_0 \in \Lambda$, then the above lemma gives that m is even. Furthermore, since $\wp(z) - \wp(z_0)$ has a zero at z_0, and since $\wp$ is also even, $\mathrm{Ord}_{z_0}(\wp) = 2$. Then,

$$f(z)\left(\wp(z) - \wp(z_0)\right)^{-\frac{m}{2}}$$

does not have a zero at z_0. Since $\wp(z) - \wp(z_0)$ has a double pole at every lattice point of Λ, we see that the $\mathrm{Ord}_0(f)$ increases by m after dividing out the zero at z_0. Similarly, if $2z_0 \notin \Lambda$, then both $f(z)$ and $\wp(z) - \wp(z_0)$ have zeroes at $\pm z_0$. If $m = \mathrm{Ord}_{z_0}(f)$, we see that

$$f(z)\left(\wp(z) - \wp(z_0)\right)^{-m}$$

no longer has zeroes at z_0 and $-z_0$; just as in the previous case, $\mathrm{Ord}_0(f)$ increases by $2m$ due to the double poles of $\wp(z) - \wp(z_0)$ at lattice points.

Since f only has finitely many zeroes, we may repeat this process to get a sequence of points $z_0, \ldots, z_n$ and integer exponents $m_0, \ldots, m_n$ such that the product

$$f(z)\prod_{j=0}^{n}\left(\wp(z) - \wp(z_j)\right)^{m_j}$$

does not have any zeroes in $\mathbb{C} \setminus \Lambda$. We may repeat this argument on the poles of f to get a sequence $w_0, \ldots, w_k$ and integer exponents $l_0, \ldots, l_k$ such that the product

$$f(z)\prod_{j=0}^{n}\left(\wp(z) - \wp(z_j)\right)^{m_j}\prod_{j=0}^{k}\left(\wp(z) - \wp(w_j)\right)^{l_j}$$

does not have any zeroes or poles in $\mathbb{C} \setminus \Lambda$. But this is still an elliptic function, and so the number of poles must equal the number of zeroes on its fundamental domain. As there is only one point in the fundamental domain that could be a pole or zero, we conclude that this product has no zeroes or poles anywhere and conclude that it must be constant. As such, if f is even, it is expressible as a rational function of $\wp$, and this gives the claim. □

5 Some Applications of $\wp$

Our analytic study of elliptic functions motivated our construction of $\wp$ — it is, in a way, the "simplest" elliptic function in that it has exactly one double pole at each lattice point. Further investigation of its properties led us to the result that $\wp$ generates every elliptic function for a given lattice.

Returning to our original motivation of elliptic integrals, it can be shown using Laurent series that $\wp$ obeys the differential equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, where g_2 and g_3 depend on the lattice Λ [5]; applying the holomorphic lifting lemma appropriately allows us to state $\wp' = \sqrt{4\wp^3 - g_2\wp - g_3}$, and after some manipulation, can be used to show that $\wp$ is the inverse of an elliptic integral [7]. As such, $\wp$ sees many applications in a wide range of physics problems, including the motion of a pendulum and the way light travels in general relativity [1].

A more surprising relationship is that of $\wp$ with the study of elliptic curves, which are a special type of projective curve. Each lattice is associated with the elliptic curve $E(\Lambda)$, which is given by

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$$

Treating $\mathbb{C}/\Lambda$ as a Riemann surface with its natural group structure and quotient topology then allows us to write the map

$$\begin{array}{rrcl} \varphi : & \mathbb{C}/\Lambda & \longrightarrow & E(\Lambda) \\ & z & \longmapsto & \big(\wp(z), \wp'(z), 1\big) \end{array}$$

with $0 \mapsto (0,1,0)$. In particular, the addition formula Theorem 15 makes φ a group homomorphism, and it can then be shown that φ is actually an isomorphism. In fact, *every* elliptic curve is isomorphic $E(\Lambda)$ for an appropiate choice of Λ [5].

Elliptic curves are applicable to a surprisingly wide variety of seemingnly unrelated fields and problems, such as factorising large numbers, cryptography, the sphere packing problem, and even the infamous Fermat's Last Theorem [4]. However, directly understanding the algebraic structures of elliptic curves is difficult (citation needed), and using the isomorphism with $\wp$ gives us a topological understanding of the structure of elliptic curves — every elliptic curve is topologically isomorphic to a torus [6].

This concludes our rather brief study of $\wp$ and elliptic functions. The early study of elliptic integrals gave rise to the concept of elliptic functions, which were later connected to doubly periodic complex functions by Abel. Using the machinery of complex analysis allowed for powerful statements on the behaviour of these elliptic functions, namely in Liouville's theorems. One of the most "well-behaved" elliptic functions is $\wp$. We showed that $\wp$ and its derivative $\wp'$ generated the entire field of elliptic functions (with respect to a fixed lattice, of course), and we briefly touched upon the importance of $\wp$ to the study of physics and elliptic curves. This only scratches the surface of the importance of elliptic functions, and one could spend a lifetime studying their various applications.

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