BROWNIAN MOTION

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Abstract. Brownian motions have been studied extensively in probability theory for their wide applicability to physical sciences and economics. It was first described by Robert Brown in his research of the motion of pollen particles suspended in water, and he investigated the motion in other fine particles suspended in other solvents. Nobody knew how to explain the motion until the work of Einstein and Smoluchowski around the turn of the century, and their explanation was fiercely contended. Eventually, Brownian motion became a staple of physics, chemistry, and even economics [2]. It was also an interest of statisticians and probability theorists during the 20th century; the study of stochastic processes and stochastic calculus had many leaps and developments, and this complemented the applications of Brownian motion in other fields [3]. Here, we give a brief introduction to the world of stochastic calculus and the role of Brownian motions, and we develop some surprising connections to harmonic and complex analysis by providing some alternate proofs to famously well-known theorems.

Contents

1. Introduction

We assume the reader has basic knowledge of measure theory and measure-theoretic probability theory, but in order to make this as self-contained as possible, we re-define some concepts and entities, potentially causing some overlap.

For the remainder of the paper, unless otherwise noted, let $(\Omega, \mathcal{F}, P)$ be a probability space and all random variables be defined on this space.

Definition 1.1 (Stochastic Process)**.** *Let (S, Σ) be a measurable space. A stochastic process is a collection of random variables $X_t : \Omega \to S$ indexed by $t \in T$. Often we take T as a subset of $\mathbb{R}$ or $[0, \infty)$ and we consider this index set to be time. This is denoted as $X_t(\omega), X(t, \omega), X_t$ or $X(t)$ with the latter two if Ω is clear.*

Throughout our discussion, we'll primarily be using $B(t)$, especially in our discussion of stochastic calculus. We will sometimes use B_t for clarity. With this essential definition, we can now define a Brownian Motion.

Definition 1.2 (Brownian Motion)**.** *A d-dimensional Brownian motion is a stochastic process $B_t : \Omega \to \mathbb{R}^d$ indexed on $[0, \infty)$ with the following properties*

i) Independent increments: For any finite sequence $t_0 < \cdots < t_n$, the increments $B_{t_{i-1}} - B_{t_i}$ are independent for $i = 0, \ldots, n$ where t_i lie in our index set.

ii) Stationary: For any pair $s, t \geq 0$ and event $A \in \mathcal{F}$,

$$P(B_{s+t} - B_s \in A) = \int_A \frac{1}{(2\pi t)^{d/2}} \exp\left(\frac{-|x|^2}{2t}\right) dx.$$

iii) The parameterization function $t \mapsto B_t$ is continuous almost surely.

We say a Brownian Motion is standard if $B_0(\omega) = 0$.

Property (ii) can also be thought of as requiring $B_{s+t} - B_t$ to be normally distributed with zero mean and variance tI where I denotes the $d \times d$ identity matrix, which we denote $B_{s+t} - B_t \sim N(0, tI)$. Moreover, observe that the RHS is independent of s. Alternatively, one may use independence of s and having zero mean as a replacement for property (ii). Then by using the Central Limit Theorem, we can derive (ii) again but with variance $t\sigma^2$ giving two equivalent definitions of Brownian Motion. It can be hard to gather intuition with this definition in higher dimensions; one can equivalently consider a d-dimensional Brownian motion to be a d-tuple of independent 1-dimensional Brownian motions, and this perhaps gives somewhat clearer insight into how high-dimensional Brownian motions can behave.

Brownian Motion itself can be interpreted in two ways. As a stochastic process, we can treat $\{B_t(\omega)\}$ as a sequence of random variables from $\Omega \to \mathbb{R}^d$ indexed by time. If we fix some $\omega \in \Omega$, we can consider $B_t(\omega)$ as a function of time and analyze it as the evolution of a path of say, a gas particle, and denote it as $B(t) : [0, \infty) \to \mathbb{R}^d$. The latter obviously has applications in the physical sciences or in financial mathematics. Though not particularly rigorous, drawing physical parallels to such scenarios often proves helpful in understanding general ideas presented in theorems and proofs in the following sections.

2. Basic Properties and Construction

It is not immediately clear that the conditions we impose in Definition 1.2 give rise to an actual product. For finite or even discrete sets of times, the second condition becomes significantly more manageable, and we can utilize this fact to build a continuous version by allowing our discrete time points to become denser and denser. We follow Lévy's construction as in [5] by finding a sequence of functions that obey this property on dyadic points and extending this to an interval before extending it to the positive reals. We require one quick lemma before tackling the construction:

Lemma 2.1 (Borel-Cantelli)**.** *Let $\{A_n\}$ be a sequence of events from $\mathcal{F}$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then*

$$P\left(\bigcap_{m=1}^{\infty} \bigcup_{n\geq m} A_n\right) = P\left(\limsup_{n\to\infty} A_n\right) = 0.$$

In fact, if the A_n's are independent in P, then

$$\sum_{n=1}^{\infty} P(A_n) = \infty \text{ if and only if } P\left(\limsup_{n\to\infty} A_n\right) = 1.$$

We omit the proof since it is an elementary exercise in basic measure theory. With this lemma, we are now able to construct a Brownian motion.

Theorem 2.2. *Standard Brownian Motion $B = \{B_t = B(t) : t \geq 0\}$ exists.*

Proof. Fix a probability space $(\Omega, \mathcal{F}, P)$. We first show existence on $[0, 1]$. Let

$$D_n := \{k/2^n : k \in [0, 2^n] \cap \mathbb{N}\}.$$

The idea is to first construct B on dyadic points and then linearly extend. Let $D := \bigcup_{n\geq 0} D_n$, which is countable, and define a sequence $\{Z_t : t \in D\}$ of independent, standard, and normally distributed random variables on $(\Omega, \mathcal{F}, P)$. Let $B(0) := 0$ and $B(1) := Z_1$. Clearly, $B(1) - B(0) \sim N(0, 1)$. For each $n \in \mathbb{N}$, we define the random variable $B(d), d \in D_n$ such that

- (i) For all $r < s < t$ in D_n, the random variable $B(t) + B(s) \sim N(0, t - s)$ and is independent of $B(s) - B(r)$.
- (ii) The vectors $(B(d) : d \in D_n)$ and $(Z_t : t \in D \setminus D_n)$ are independent.

We have already have shown this for $D_0 = \{0, 1\}$ so suppose we have a construction for $n - 1$. Then define $B(d)$ for $d \in D_n \setminus D_{n-1}$ by

$$B(d) = \frac{B(d - 2^{-n}) + B(d + 2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}}.$$

The first summand is a linear interpolation of the values of B at the neighboring points of d in D_{n-1}. Thus $B(d)$ is independent of $(Z_t : t \in D \setminus D_n)$ and (ii) is satisfied.

Moreover as $\frac{1}{2}[B(d - 2^{-n}) + B(d + 2^{-n})]$ depends only on $(Z_t : t \in D_{n-1})$, it is independent of $Z_d/2^{(n+1)/2}$. By our inductive hypothesis, both terms are normally distributed with zero mean and variance $2^{-(n+1)}$ therefore

$$B(d) - B(d-2^{-n}) = \frac{-B(d-2^{-n}) + B(d+2^{-n})}{2} + \frac{Z_d}{2 \cdot 2^{(n-1)/2}}$$
$$B(d+2^{-n}) - B(d) = \frac{B(d+2^{-n}) - B(d-2^{-n})}{2} - \frac{Z_d}{2 \cdot 2^{(n-1)/2}}$$

are independent and normally distributed with zero mean and variance 2^{-n}.

Indeed, all increments $B(d) - B(d-2^{-n})$ for $0 \neq d \in D_n$ are independent. Since the vector of these increments is Gaussian, it suffices to show pairwise independence by Isserlis' theorem. By the above, the pairs $B(d) - B(d-2^{-n})$ and $B(d+2^{-n}) - B(d)$ are independent with $d \in D_n \setminus D_{n-1}$. Consider the case when the increments are over intervals separated by $d \in D_{n-1}$. Choose such a $d \in D_j$ where j is minimal so the intervals are contained in $[d - 2^{-j}, d]$ and $[d, d+2^{-j}]$. By induction, the increments over these intervals of length 2^{-j} are independent. The increments over intervals of length 2^{-n} are constructed from independent increments $B(d) - B(d-2^{-j})$ and $B(d+2^{-j}) - B(d)$ respectively using a disjoint set of variables $(Z_t : t \in D_n)$. Thus, they are independent and implies (i), closing the induction.

Having chosen the values on dyadic points, we can linearly interpolate. Define

$$F_0(t) := \begin{cases} Z_1, & t = 1 \\ 0, & t = 0 \\ Z_1 t, & 0 < t < 1 \end{cases}$$

and for each $n \geq 1$,

$$F_n(t) := \begin{cases} 2^{-(n+1)/2} Z_t, & t \in D_n \setminus D_{n-1} \\ 0, & t = 0 \\ \text{linear} & \end{cases}$$

where $F_n(t)$ is linear between consecutive points in D_n. We then have for $d \in D_n$,

$$B(d) = \sum_{j=0}^{n} F_j(d) = \sum_{j=0}^{\infty} F_j(d).$$

We claim the resulting process is continuous on $[0,1]$. We start by showing the series

$$B(t) = \sum_{n=0}^{\infty} F_n(t) \tag{1}$$

is uniformly convergent. By the Chernoff bound on Gaussian tails, we have for large c and n,

$$P(|Z_d| \geq c\sqrt{n}) \leq e^{-c^2 n/2}.$$

Let A be the event that there exists $d \in D_n$ with $|Z_d| \geq c\sqrt{n}$ so the series

$$\sum_{n=0}^{\infty} P(A) \leq \sum_{n=0}^{\infty} \sum_{d \in D_n} P(|Z_d| \geq c\sqrt{n}) \leq \sum_{n=0}^{\infty} (2^n + 1) \exp\left(\frac{-c^2 n}{2}\right)$$

converges for $c \geq \sqrt{2 \log 2}$. Fix such a c, then by Borel-Cantelli, there exists a random N, almost surely finite, such that for all $n \geq N$ and $d \in D_n$, we have $|Z_d| < c\sqrt{n}$. So for all $n \geq N$,

$$\|F_n\|_\infty < \frac{c\sqrt{n}}{2^{n/2}},$$

note that we do not need to specify which infinity norm we use here since F_n is continuous so the L^∞ and the sup-norm are equivalent. This implies that the series (1) converges uniformly almost surely on $[0, 1]$. Further, the increments have the correct distribution on the dense set $D \subset [0, 1]$ and therefore on the whole interval.

To extend this from the unit interval to $[0, \infty)$, we pick a sequence $B_0, B_1, \ldots$ of independent random variables valued on $C[0, 1]$ with this distribution and define $\{B(t) : t \geq 0\}$ by gluing these parts to make a continuous function. □

Now that we know Brownian motion is a well-defined object, we can begin introducing some of its properties and lay the foundations of Stochastic Calculus, which we give an introduction to in Section 3.

For the sake of completeness, we give the following definition:

Definition 2.3 (Convergence in Probability)**.** *We say a sequence of random variables $\{X_n\}$ converges in probability to a random variable X if for all $\varepsilon > 0$,*

$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0.$$

Oftentimes, we just abuse notation and use the lim operator when refering to convergence in probability.

Definition 2.4 (Quadratic Variation)**.** *Let X_t be a stochastic process indexed on the non-negative reals. The quadratic variation is the process*

$$[X]_t := \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \left(X_{t_i} - X_{t_{i+1}}\right)^2$$

where $0 = t_0 \leq \cdots \leq t_n = t$ is a partition of $[0, t]$ and $\|P\|$ denotes the mesh size. Further, this convergence is in probability.

For familiar functions such as continuous and differentiable functions, they will have zero quadratic variation. We can extend this to more than one process by defining the covariance.

Definition 2.5 (Covariation). *Let X and Y be two stochastic processes. The covariation of X and Y is*

$$[X, Y]_t := \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \left(X_{t_i} - X_{t_{i+1}} \right) \left(Y_{t_i} - Y_{t_{i+1}} \right)$$

where we use the same notation as Definition 2.4.

Lemma 2.6 (Basic Properties of Brownian Motion). *Let $B(t)$ indexed on the non-negative reals be a standard Brownian Motion. Then we have the following:*

a) $B(t)$ is nowhere differentiable.
b) $B(t)$ has finite quadratic variation.
c) (Scaling Invariance) Let $a > 0$. The process $\{\frac{1}{a} B(a^2 t)\}$ is also a standard Brownian Motion.
d) (Time Inversion) The process $\{tB(1/t)\}$ is a standard Brownian Motion.

For the proofs, we refer the reader to [4] and [5].

One important property of random variables is that of the Markov property, which Brownian motion satisfies by construction.

Theorem 2.7 (Markov Property). *Let $\{B(t) : t \geq 0\}$ be a Brownian motion starting from $x \in \mathbb{R}^d$. For $s > 0$, the process $\{B(t+s) - B(s) : t \geq 0\}$ is a standard Brownian motion and independent of $\{B(t) : t \in [0, s]\}$.*

The Markov property for random variables states that the process is "memory-less" in that the evolution depends only on the present state at t and not for any previous time $s < t$. As we will see in Section 3, there is a way to refine this into an exact estimate of future expected values but for now, Markovian processes only give guidance on how to make future predictions.

Definition 2.8 (Transience and Recurrence). *Brownian motion $\{B(t) : t \geq 0\}$ is called*

- *transient if $|B(t)| \to \infty$ a.s.*
- *point recurrent if a.s., for every $x \in \mathbb{R}^d$ and $\varepsilon > 0$, there exists a sequence t_n diverging to ∞ such that $B(t_n) = x$ for all n.*
- *neighborhood recurrent if a.s., for every $x \in \mathbb{R}^d$ and $r > 0$, there exists a sequence t_n diverging to ∞ such that $B(t_n) \in B(x, r)$ for all n.*

We will demonstrate later that Brownian motion in one dimension is point recurrent, is neighborhood recurrent in two dimensions, and transient for higher dimensions. Going by intuition and consider a random walk, this seems plausible: given an infinite amount of time, an infinitely drunk tightrope walker on an infinite tightrope will stumble over every section of the tightrope an infinite amount of time. To be rigorous, however, we need to establish the machinery of stochastic calculus. Afterwards, we'll find ourselves at a surprising crossroads with harmonic functions, and in the two-dimensional case, an interesting relationship with complex analysis.

3. Stochastic Calculus

Just as the existence of Brownian motion was easiest in the discrete case and carefully extended to the continuous formulation, we'll start by describing ideas about discrete stochastic processes that are naturally extensible to continuous stochastic processes. A common and intuitive example is the "accumulation" of a random variable over time. The typical example is a stochastic process $\{B(n) : n \in \mathbb{N}\}$ that describes the amount of money a gambler wins on day n; naturally, the amount of money the gambler earns on day n might be given by

$$\sum_{i=1}^{n} \mathbb{E}\left[B(i) - B(i-1)\right]$$

One might even be interested in sums of $f\left(B(i)\right) - f\left(B(i-1)\right)$, where f describes some other quantity determined by the winnings each day (say the amount of weight the gambler has gained or lost). If B were not a random variable, the theory of calculus helps us refine this into an integral. We would be tempted to write something like:

$$f\left(B(t)\right) - f\left(B(a)\right) = \int_a^t f'\left(B(s)\right) B'(s) ds$$

The big pitfall, of course, is the fact that B is almost never differentiable, as seen in Lemma 2.6. This is because we naïvely tried writing

$$\frac{d}{dt} f\left(B(t)\right) = f'\left(B(t)\right) B'(t)$$

or, in the language of differentials,

$$df = f'\left(B(t)\right) dB$$

To motivate this section, we will develop some of the theory of Itô Calculus, which adjusts the above differential and gives meaning to integrals over dB. Itô Calculus is a deep field of stochastic probability theory and stochastic calculus, and so some calculations or proofs have been omitted to communicate the most important aspects of the theory.

3.1. **Itô Calculus.**

In order to address these ideas, we need to equip probability spaces with additional σ-algebra structure that evolves with time. One may think of this heuristically as being "the information from the beginning to time t", or a collection of all the outcomes up to a certain moment.

Definition 3.1 (Filtration)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration on this space is a collection of σ-algebras $\{\mathscr{F}(t) : t \geq 0\}$ such that $\mathscr{F}(s) \subseteq \mathscr{F}(t)$ for $s \leq t$.*

Observe that immediately we have $\mathscr{F}(s) \subseteq \mathcal{F}$ for all t. Indeed, if $\mathcal{F}$ is the set of all possible events, then the collection of outcomes will be a subset.

Definition 3.2 (Adapted)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}(t) : t \geq 0\}$ and $\{X_t : t \geq 0\}$ be a stochastic process on this space. The process is adapted if for each t, $X_t = X(t)$ is measurable with respect to $\mathscr{F}(t)$.*

Suppose $B(t)$ is a Brownian motion, suppose $\mathscr{F}(t)$ is a filtration of Ω, and suppose $f(t,\omega)$ is an adapted stochastic process with respect to $\mathscr{F}(t)$ and obeys the additional condition that $\int_a^b \mathbb{E}\left[|f(t)|^2\right] dt < \infty$. We will construct a stochastic integral that gives meaning to

$$\int_a^b f(t)dB(t)$$

Following [3], we'll construct this integral by first considering simple "step stochastic processes" before approximating arbitrary stochastic processes. Then, we will define the above integral by taking the limit of these approximations. This is quite reminiscent of how the Lebesgue integral is typically constructed.

Definition 3.3 (Step Stochastic Processes)**.** *$a = t_0 < t_1 < \ldots < t_n = b$ be a partition of the interval $[a, b]$. $f(t, \omega)$ is a step stochastic process if it is of the form*

$$f(t,\omega) = \sum_{i=1}^{n} \xi_{i-1}(\omega) 1_{[t_{i-1},t_i)}(t)$$

where 1 denotes the indicator function on a specified domain and ξ_{i-1} is a measurable function on $\mathscr{F}(t_{i-1})$.

We call such an f "square-integrable". In order for f to be square-integrable in the above sense, we necessarily need $\mathbb{E}[\xi_{i-1}^2] < \infty$. Much like step functions, one may think of step stochastic processes as one that instantly transitions through a sequence of random variables over time. The natural way to define the integral for a step stochastic process as above is therefore

$$\int_a^b f(t)dB(t) := \sum_{i=1}^{n} \xi_{i-1} \left(B(t_{i-1}) - B(t_i)\right)$$

Much like how Lebesgue-integrable functions can be approximated by step functions, we have an important lemma that establishes an analogue for stochastic processes:

Lemma 3.4. *For any stochastic process f, there exists a sequence of square-integrable step stochastic processes f_n such that*

$$\lim_{n\to\infty} \int_a^b \mathbb{E}\left[|f(t) - f_n(t)|^2\right] dt = 0$$

The full proof is heavily computational, but this claim is not trivial in the slightest. We will provide a proof sketch that omits the computation but still aims to describe the core argument behind this claim.

Proof. Case 1: f is continuous

This is the nicest case, and much of Lebesgue measure theory applies here. Continuity of f makes it nice enough that the observation that

$$\lim_{n\to\infty} \mathbb{E}\left[|f(t) - f_n(t)|^2\right] = 0$$

pointwise almost surely implies the claim via dominated convergence.

Case 2: f is bounded

Continuity of f is a relatively strict condition to impose on stochastic processes. When f is not continuous but still bounded, we simplify the problem by making a sequence of continuous "smeared out" versions of f. Similar techniques are not uncommon throughout analysis, and the Weierstrass approximation theorem gives the existence of arbitrarily fine polynomial approximations to continuous functions under certain conditions. Convolution with a smooth bump function is another way of smoothing out a function, though both these cases make smooth approximations to continuous functions. Here, the way we elect to "smear out" f is as follows:

$$g_n(t,\omega) = \int_0^{n(t-a} e^{-s} f\left(t - \frac{s}{n}, \omega\right) ds = \int_0^t ne^{-n(t-u)} f(u,\omega) du$$

The larger n is, the sharper the smudges get: the only meaningful contributions of f to the integral occur when its time component is close to t. These are continuous, and applying the first case to the g_n's with some careful manipulation of limits gives the claim.

Case 3: f is unbounded

For the last class of functions, we may take "cutoffs" of f as follows:

$$g_n(t) = \begin{cases} f(t) & |f(t)| \leq n \\ 0 & f(t) > n \end{cases}$$

Applying case 2 to the g_n's alongside a standard application of dominated convergence produces the claim. □

Though tedious, this establishes the existence of the stochastic integral, which is as fundamental to stochastic and Itô calculus as the Lebesgue integral is to measure theory. Stochastic integrals are actually a much broader class of integrals, and this particular integral is named the Itô integral after Kiyoshi Itô. It also has an isometric property, which can be demonstrated through some calculations and by using the basic properties of Brownian motion.

Theorem 3.5 (Itô Integrals). *For any square-integrable f obeying the conditions outlined at the beginning of this section and Brownian motion B, the Itô integral of f with respect to B is a random variable given by $\int_a^b f(t)dB(t)$, which is defined as the limit from above. Its expected value is 0, and*

$$\mathbb{E}\left[\left|\int_a^b f(t)dB(t)\right|^2\right] = \int_a^b \mathbb{E}\left[|f(t)|^2\right] dt$$

Corollary 3.6. *If $f(t)$ is a continuous $\mathscr{F}$-adapted stochastic process on $[a,b]$, then*

$$\int_a^b f(t)dB(t) = \lim_{\mu(P)\to 0} \sum_{i=1}^n f(t_{i-1})(B(t_i) - B(t_{i-1}))$$

in probability, where $P = \{t_0, \ldots, t_n\}$ is a partition of $[a,b]$ and $\mu(P)$ is its mesh size.

This corollary is very reminiscent of the Riemann integral, and it follows quite readily from continuity of f together with the step approximations from before.

3.2. **The Itô Formula.** There are two issues for the moment. First, f only depends on t, so we do not yet know how to integrate functions that depend on $B(t)$ (or even both t and $B(t)$). Second, while we have defined the Itô integral, we do not yet know what value it holds. Itô's formula is a central result in stochastic calculus because it addresses both of these — it provides both a concrete value for the Itô integral from above and gives us a meaningful way to take the differential $df(B(t))$.

We'll follow how [3] develops the general Itô formula. We'll first develop the Itô formula in a simplified setting, then we'll observe how it arguments and techniques generalize to other settings. As with before, several lengthy computations will be omitted for clarity's sake.

Theorem 3.7. *Let $f(t)$ be continuously twice differentiable. Then,*

$$f(B(t)) - f(B(a)) = \int_a^t f'(B(s))dB(s) + \frac{1}{2}\int_a^t f''(B(s))ds$$

Written in differential form,

$$df(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt$$

We'll remark that the first integral is taken as an Itô integral, as described in the previous subsection. However, the second integral is an ordinary Riemann or Lebesgue integral. This formula also seems to suggest using a second-order Taylor expansion, and this is indeed the way in which it is derived. Again, we'll provide an outline, leaving strenuous computations out for brevity's sake.

Proof. First, let $P = \{t_0, \ldots, t_n\}$ be any partition of $[a, t]$. Rewrite the difference on the left as a telescoping series:

$$\begin{aligned} &f(B(t)) - f(B(a)) \\ &= \sum_{i=1}^{n} (f(B(t_i)) - f(B(t_{i-1}))) \end{aligned}$$

Apply a second-order Taylor expansion. For some $b_1, \ldots, b_n$, where for all i, b_i lies between $B(t_i)$ and $B(t_{i-1})$, we have that

$$= \sum_{i=1}^{n} \left(f'(B(t_{i-1}))(B(t_i) - B(t_{i-1})) + \frac{1}{2}f''(b_i)(B(t_i) - B(t_{i-1}))^2 \right)$$

By Corollary 3.6, the first term in the summand converges in probability to $\int_a^t f'(B(s))dB(s)$ as the mesh size vanishes. The second term resembles the second integral in the formula, but the issue is that the differential-resembling term is in B rather than t; moreover, it's squared.

Fortunately, we may split the sum

$$\sum_{i=1}^{n} f''(b_i)\left(B(t_i)-B(t_{i-1})\right)^2 = \sum_{i=1}^{n}\left(f''(b_i)-f''(B(t_i))\right)\left(B(t_i)-B(t_{i-1})\right)^2 + \sum_{i=1}^{n} f''(B(t_i))\left(\left(B(t_i)-B(t_{i-1})\right)^2-(t_i-t_{i-1})\right) + \sum_{i=1}^{n} f''(B(t_i))\left(t_i-t_{i-1}\right)$$

It can then be shown that the first two sums here vanish almost surely as the mesh size gets arbitrarily small [3] [8]. In short, the reason is that the in the first term, the difference of $f''(b_i) - f''(B(t_i))$ almost surely vanishes uniformly on $[a,t]$ while the squared difference in B resembles $t_i - t_{i-1}$. For this very same reason, boundedness of f'' alongside the estimate on the squared difference of B makes the second sum almost surely vanish as the mesh size decreases to 0. The last sum here is just a standard Riemann sum, allowing us to conclude that

$$\frac{1}{2}\sum_{i=1}^{n} f''(b_i)\left(B(t_i)-B(t_{i-1})\right)^2 \to \frac{1}{2}\int_a^t f''(B(s))ds$$

as $\mu(P) \to 0$, producing the desired result. □

By applying similar techniques and gruelling over progressively worse and worse computations, we can generalize these results to a class of stochastic processes known as Itô processes.

Definition 3.8 (Itô Process)**.** *Let $\mathscr{F}(t)$ be a filtration. X_t is an Itô process if there exist f, g adapted square-integrable stochastic processes on $[a,b]$ with respect to $\mathscr{F}(t)$ and X_a $\mathscr{F}(a)$-measurable such that*

$$X_t = X_a + \int_a^t f(s)dB(s) + \int_a^t g(s)ds$$

for $t \in [a,b]$.

We will state without proof the generalization of this result to Itô processes as above. Both will be necessary in unpacking higher-dimensional Brownian motions and understanding their properties. The proof of this is not much different from above; the argument still consists of decomposing a telescoping series with a Taylor expansion, then identifying which subsums vanish and which can be identified with the corresponding integrals in the theorem.

Theorem 3.9. *Let $\theta(t,x)$ be continuously differentiable in t and twice continuously differentiable in x. Let $X(t)$ be an Itô process as described above. Then, $\theta(t, X(t))$ is* *also an Itô process given by*

$$\theta(t, X(t)) = \theta(a, X(a)) + \int_a^t \frac{\partial\theta}{\partial x}(s, X(s)) \cdot f(s)dB(s) + \int_a^t \left(\frac{\partial\theta}{\partial t}(s, X(s)) + \frac{\partial\theta}{\partial x}(s, X(s)) \cdot g(s) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(s, X(s))f(s)^2 \right) ds$$

This generalization is crucial to understanding Brownian motion in higher dimensions, as we will shortly see. Another generalization we will need is the extension of this formula to multiple dimensions; as one might expect though, this can be done by handling each component independently, and there is an extra contribution from mixed partials. In differential form, if $\theta(t, x_1, \ldots, x_n)$ has one continuous derivative in t and is twice continuously differentiable in $x_1, \ldots, x_n$ (including mixed partials), then we have for an independent set of Itô processes $X_1(t), \ldots, X_n(t)$,

$$d\theta(t, X_1(t), \ldots, X_n(t) = \frac{\partial\theta}{\partial t}dt + \sum_{i=1}^n \frac{\partial\theta}{\partial x_i}dX_i + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2\theta}{\partial X_i\partial X_j}dX_idX_j$$

An important detail to note is that whereas this multidimensional formulation has dX_idX_j terms in it, the previous one did not. Furthermore, we recall that in the proof of the simplest case, we showed that the sum

$$\sum_{i=1}^n g(b_i)\left((B(t_i) - B(t_{i-1}))^2 - (t_i - t_{i-1})\right)$$

almost surely vanishes as the partition became arbitrarily fine. Heuristically, this tells us that $dB^2 = dt$; this makes sense from the definition of Brownian motion, and this is indeed the case. In multiple dimensions, we identify $dB_i^2 = dt$ for all i. However, one additional piece of subtlety arises from the mixed differentials. It can be shown through computation in expanding the aforementioned sums that $dB_idB_j = d\left[B_i(t), B_j(t)\right]_t$. That is, dt has been scaled by the quadratic covariance of B_i and B_j with respect to t. When $i = j$, $\left[B_i(t), B_i(t)\right]_t = t$, and this recovers the original generalization in one dimension.

Taking $n = 1$ reduces this formula back down to the previous case. Expressing this in integral form can be done, but it is cumbersome to write out in most cases.

3.3. **Levy's Characterization Theorem.** Although we have constructively shown the existence of Brownian motions and their applicability in stochastic calculus, we don't have a good bearing for what Brownian motions look like. Indeed, checking the conditions in their definition can be rather difficult when presented with an arbitrary stochastic process. Levy's characterization theorem, as the name suggests, uses the machinery we set up earlier to find a way to identify Brownian motions.

Definition 3.10 (Martingale)**.** *If X_t is a stochastic process adapted to a filtration $\mathscr{F}$, if $\mathbb{E}\left[X_t\right] < \infty$ almost surely for all t, and if $s \leq t$ implies that $\mathbb{E}\left[X_t \mid s\right] = X_s$ almost surely, then X_t is a martingale.*

Intuitively, a martingale is another stochastic process that "forgets" about the past — no outcome is influenced by the events that occurred before it. Comparing this with Thm 2.7, we see that both characteristics (Markov and Martingale) disregard all past states and use only the infomation of the current state to predict the future. However for a martingale process, the best estimate for future values is exactly the present value.

Lemma 3.11. *Let $M(t)$ be a martingale for $t \in [a, b]$, and let $f(t)$ be any square-integrable function over $[a, b]$. Then,*

$$\int_a^t f(s) dM(s)$$

is a martingale.

Theorem 3.12 (Levy's Characterization Theorem)**.** *Let $X(t)$ be a stochastic process. $X(t)$ is a Brownian motion if and only if there exists a probability measure Q and filtration $\mathscr{F}_t$ such that $X(t)$ is a continuous martingale with respect to $\mathscr{F}_t$ and Q, if $Q(X(0) = 0) = 1$, and $[X(t)]_t = t$ almost surely for all t with respect to Q.*

To stipulate that $X(t)$ is a martingale is not enough for it to be a Brownian motion. Without going too deep on a tangent, one can offset a Poisson process (a generalization of the Poisson distribution in the language of stochastic processes) by a function of t and produce a discontinuous stochastic process that fits all of the above criteria, yet isn't a Brownian motion [3].

For the proof, we direct the reader to [3]. In brief, the "only if" direction follows readily from the definition and properties of a Brownian motion. The forward direction relies on finding a clever application of Itô's formula and performing many routine calculations. The argument itself does not contain a significant amount of nuance (despite this theorem being quite powerful), and so we elect to omit the proof.

4. Applications to Complex Analysis

With the machinery of stochastic calculus, we are now able to begin moving towards the proof of Liouville's Theorem but from the perspective of Brownian Motion. The purpose is to give an alternative derivation and provides a good exercise for the probabilist reviewing their complex analysis. This material follows [5] and [8] but with added commentary.

Recall from Section 2 that one important characteristic of Brownian Motion is that the future states, given the present, will only depend on the present; this was called being a Markov process. For Brownian motion, we can generalize the "present" to be in terms of a random variable rather than a fixed time. This is called the strong Markov property and immediately implies Theorem 2.7.

Suppose we have a Brownian Motion $\{B(t) : t \geq 0\}$, then we can define a filtration $\{\mathscr{F}^0(t) : t \geq 0\}$ by letting

$$\mathscr{F}^0(t) = \sigma\{B(s) : s \in [0, t]\}.$$

This is clearly a σ-algebra and makes Brownian Motion adapted to the filtration. Intuitively, one may think of this as all the information gathered from observing the process up to some time t.

From Theorem 2.7, we know $\{B(t+s) - B(s) : t \geq 0\}$ is independent of $\mathscr{F}^0(s)$ so we can improve the filtration and define a larger σ-algebra by

$$\mathscr{F}^+(s) = \bigcap_{t>s} \mathscr{F}^0(t).$$

This is still a filtration and satisfies $\mathscr{F}^0(s) \subset \mathscr{F}^+(s)$. Intuitively, as $\mathscr{F}^+$ is larger, this is like getting an infinitely small glance into the future.

Theorem 4.1. *For all $s \geq 0$, the process $\{B(t+s) - B(s) : t \geq 0\}$ is independent of $\mathscr{F}^+(s)$.*

Proof. This follows from continuity of the increments and Theorem 2.7. □

4.1. Strong Markov Property.

Definition 4.2 (Stopping Time)**.** *A random variable $T : \Omega \to [0, \infty]$ defined on a probability space with filtration $\{\mathscr{F}(t) : t \geq 0\}$ is called a stopping time with respect to the filtration if $\{T(\omega) \leq t\} \in \mathscr{F}(t)$ for all $t \geq 0$.*

Consider a one-dimensional Brownian motion and let $a \in \mathbb{R}$. Let $T := \inf\{t \geq 0 : B(t) > a\}$. Then this is a stopping time and can be thought of as corresponding to the rule "stop when B passes a."

With Brownian motion, we typically consider stopping times with respect to the filtration $\{\mathscr{F}^+\}$ constructed above because the filtration is larger, giving us access to more stopping times. Further observe that if we consider a small time shift into the future, then

$$\bigcap_{\varepsilon>0} \mathscr{F}^+(t+\varepsilon) = \mathscr{F}^+(t).$$

This property is called right-continuity and is one advantage $\mathscr{F}^+$ holds over $\mathscr{F}^0$.

Lemma 4.3. *Suppose a random variable $T : \Omega \to [0, \infty]$ satisfies $\{T(\omega) \leq t\} \in \mathscr{F}(t)$ for all $t \geq 0$ and $\{\mathscr{F}(t)\}$ is a right-continuous filtration, then T is a stopping time with respect to $\{\mathscr{F}(t)\}$.*

Proof. This follows from right-continuity of $\mathscr{F}$. □

Let T be a stopping time, then define the σ-algebra

$$\mathscr{F}^+(T) := \{S \in \mathcal{F} : S \cap \{T(\omega) \leq t\} \in \mathscr{F}^+(t), t \geq 0\}.$$

This represents the collection of events up to the stopping time and also immediately shows that $\{B(t) : t \leq T\}$ is $\mathscr{F}^+(T)$-measurable.

With the concept of stopping times, we can prove an important corollary to Lévy's theorem.

Corollary 4.4 (Dubins-Schwarz)**.** *Let X be a continuous local martingale with $X(0) = 0$ such that $[X]_t$ increases as $t \to \infty$. For $t \geq 0$, define stopping times*

$$T_t = \inf\{s : [X]_s > t\}$$

and a shifted filtration $\mathscr{G}(t) = \mathscr{F}(T_t)$. Then $M(t) = X(T_t)$ is a standard Brownian Motion.

Theorem 4.5 (Strong Markov Property)**.** *For every almost surely finite stopping time T, the process*

$$\{B(T+t) - B(T) : t \geq 0\}$$

is standard Brownian Motion is independent of $\mathscr{F}^+(T)$.

Proof. We first prove this for discrete stopping time T_n and then approximate T. Let

$$T_n = \frac{m+1}{2} \text{ if } T \in \left[\frac{m}{2^n}, \frac{m+1}{2^n}\right).$$

Let $B_k = \{B_k(t) : t \geq 0\}$ be the Brownian motion defined by

$$B_k(t) := B\left(t + \frac{k}{2^n}\right) - B\left(\frac{k}{2^n}\right)$$

and $B_* = \{B_*(t) : t \geq 0\}$ be the Brownian motion defined by

$$B_*(t) = B(t + T_n) - B(T_n).$$

Suppose $E \in \mathscr{F}^+(T_n)$, then for all events $\{B_* \in S\}$ (where $S \in \mathcal{F}$), we have

$$P(\{B_* \in S\} \cap E) = \sum_{k \geq 0} P(\{B_k \in S\} \cap E \cap \{T_n = k/2^n\}),$$

but as $\{B_k \in S\}$ and $E \cap \{T_n = k/2^n\}$ are independent from Theorem 4.1,

$$\begin{aligned} P(\{B_* \in S) \cap E) &= \sum_{k \geq 0} P(B_k \in S) P(E \cap \{T_n = k/2^n\}) \\ &= P(B \in S) \sum_{k \geq 0} P(E \cap \{T_n = k/2^n\}) \\ &= P(B \in S) P(E) \end{aligned}$$

where we use $P(B_k \in S) = P(B \in S)$ for all k since Brownian Motion is Markovian and hence does not depend on k. Thus, we have that B_* is a Brownian motion independent of E and hence of $\mathscr{F}^+(T_n)$.

Now let T be a general stopping time. As T_n approximates T from above, we have $\{B(s + T_n) - B(T_n) : s \geq 0\}$ is a Brownian motion independent of $\mathscr{F}^+(T_n)$, which contains $\mathscr{F}^+(t)$. So

$$B(s + t + T) - B(t + T) = \lim_{n \to \infty} B(s + t + T_n) - B(t + T_n)$$

are the increments of $\{B(x+T)-B(T) : x \geq 0\}$ and independent. Further,

$$B(s+t+T)-B(t+T) \sim N(0,s).$$

Since this is almost surely continuous, it is a Brownian Motion. As all increments can be written as

$$B(s+t+T)-B(t+T) = \lim_{n\to\infty} B(s+t+T_n)-B(t+T_n),$$

we conclude it is independent of $\mathscr{F}^+(T)$. □

Alternatively, one may write for any bounded measurable $f : C([0,\infty),\mathbb{R}^d) \to \mathbb{R}$, we have

$$\mathbb{E}[f(\{B(T+t) : t \geq 0\})|\mathscr{F}^+(T)] = \mathbb{E}_{B(T)}[f(\{\tilde{B}(t) : t \geq 0\})]$$

where the RHS is the expectation with respect to a Brownian Motion $\{\tilde{B}(t)\}$ starting at $B(T)$.

As a result of the Strong Markov Property, Brownian motion satisfies what is known as the Reflection Principle, which states that reflecting a Brownian motion at some stopping time T is still a Brownian motion. Another important consequence is that in the plane, the path of Brownian motion on $[0,1]$ has Lebesgue measure 0 almost surely. This is interesting because some continuous curves can be space-filling, but as a result of the Markov property and Reflection Property, this cannot be the case for Brownian motion. See [5] for more details. However it is dense in the plane, which we discuss below.

4.2. **The Dirichlet Problem.** Intricately connected with the concept of recurrence (Definition 2.8) for Brownian Motion is the Dirichlet problem for reasons we will see momentarily.

Recall that for $U \subseteq \mathbb{R}^d$, a harmonic function $u : U \to \mathbb{R}$ is a function that is C^2 and satisfies $\Delta u = 0$. Moreover, these functions satisfy important mean value conditions.

Lemma 4.6. *Let $U \subset \mathbb{R}^d$ and $u : U \to \mathbb{R}$ be measurable and locally bounded. Then the following are equivalent:*

a) u is harmonic on U.
b) For any ball $D = D(x,r) \subseteq U$, we have

$$u(x) = \frac{1}{\mathrm{vol}(D)} \int_{D(x,r)} u(t)\, dt.$$

c) For any ball $B = B(x,r) \subseteq U$, we have

$$u(x) = \frac{1}{\sigma_r(\partial D)} \int_{\partial D} u(y)\, d\sigma_r$$

where σ_r is the surface measure on ∂D.

We state one more theorem without proof as it is standard in any complex or harmonic analysis course and it is that of the maximum principle.

Theorem 4.7 ((Complex) Maximum Principle)**.** *Let $U \subset \mathbb{C}$ be open and connected and suppose $u : U \to \mathbb{C}$ is harmonic. Let $K \subset U$ be compact, then u must attain its maximum on the boundary of K.*

This immediately implies that if two functions are harmonic on U and agree on $\partial K \subset U$, then they are identical. Naturally, this extends to any harmonic function $u : U \to \mathbb{R}^d$.

Theorem 4.8. *Let $U \subset \mathbb{R}^d$. For any $x \in U$, let $\{B_x(t) : t \geq 0\}$ be a Brownian motion started at x, and*

$$T_x = \min\{t \geq 0 : B_x(t) \in \partial U\}$$

be the first time B_x hits the boundary of U. Let $\varphi : \partial U \to \mathbb{R}$ be measurable and such that for $u : U \to \mathbb{R}$ satisfying

$$u(x) = \mathbb{E}[\varphi(B_x(T_x))\mathbf{1}_{\{T_x<\infty\}}]$$

is locally bounded. Then u is harmonic.

Proof. Let $x \in U$, and let $D(x,r) \subseteq U$ be a ball of radius r centered at x. Let

$$T'_x = \min\{t \geq 0 : B_x(t) \in \partial D(x,r)\}$$

Then, we have by the strong Markov property that

$$\mathbb{E}\left[\phi(B_x(T))\mathbf{1}_{T<\infty} \mid \mathscr{F}^+(T'_x)\right] = u(B_x(T'_x))$$

Take the expectation of both sides. On one hand, the expectation of the left side is just

$$\mathbb{E}\left[\mathbb{E}\left[\phi(B_x(T_x))\mathbf{1}_{T_x<\infty} \mid \mathscr{F}^+(T'_x)\right]\right] = u(x)$$

because of the towering property of conditional expectations. On the other hand, the expectation of the right side is

$$\mathbb{E}\left[u(B_x(T'_x))\right] = \frac{1}{\sigma_r(\partial D(x,r))}\int_{\partial D(x,r)} u(y)d\sigma_r$$

because $B_x(T'_x)$ is uniformly distributed over $\partial D(x,r)$. Putting the two together demonstrates that u obeys the mean value property while staying locally bounded; it follows that u is harmonic on U. □

As with many questions regarding PDEs, the Dirichlet problem requires certain niceness conditions on the set in order to guarantee a solution. For our purposes, we can quantify this niceness using the Poincaré cone condition, which $U \subseteq \mathbb{R}^d$ satisfies at $x \in \partial U$ if there exists a cone V with vertex at x, opening angle $\theta > 0$, and $r > 0$ such that $V \cap B(x,r) \subset U^c$. Physically, this means that U cannot have any inward-pointing cusps (though they may point outward).

Before tackling the Dirichlet problem, we need to establish one last lemma regarding Brownian motions and cones.

Lemma 4.9. *Let $C_0(\theta)$ be a cone centered at the origin as described above, and let*

$$a = \sup_{x \in D(0,\frac{1}{2})} P(T_0(\partial D(0,1)) < T_0(C_0(\theta))$$

where $T_x(S)$ is the first hitting time of a Brownian motion starting at x on a set S. Then, if k and h are positive integers,

$$P(T_x(\partial D(z,h)) < T_x(C_z(\theta))) \leq a^k$$

whenever $|x - z| < 2^{-k}h$.

The proof can be found in [5]. The point here is that the closer a Brownian motion starts to the "tip" of $C_x(\theta) \cap D(x,r)$, the more likely you are to exit through the straight edges of the cone rather than through the arc. Together, this lets us solve the Dirichlet problem:

Theorem 4.10 (Dirichlet Problem)**.** *Let $U \subseteq \mathbb{R}^d$ be bounded and connected, and suppose that it satisfies the Poincare cone condition on its boundary. Let φ be continuous on ∂U, and let*

$$T_x = \inf\{t \geq 0 : B_x(t) \in \partial U\}$$

where B_x is a Brownian motion starting at x. Then,

$$u(x) = \mathbb{E}\left[\varphi\left(B_x(T_x)\right)\right]$$

is the unique continuous harmonic function on $\overline{U}$ with $u(x) = \varphi(x)$ on ∂U.

Proof. Uniqueness follows immediately from harmonic continuation. $u(x)$ is clearly locally bounded, for U is a bounded set, and so by Theorem 4.8 u is harmonic inside U. Of course $u(x) = \varphi(x)$ on $x \in \partial U$, so we only need to verify continuity of u near ∂U.

Let $z \in \partial U$, and let x be close to z. Let $C_z(\theta)$ be some cone as described in the Poincare cone condition. Then, by the lemma, $B_x(T_x)$ is very likely to be close to z: it is much more likely that B_x hits the cone $C_z(\theta)$ before wandering far from z, and B_x must hit ∂U before hitting the cone. Thus, $\phi(B_x(T_x))$ is almost surely close to $\phi(z)$, and it follows from continuity of ϕ that u is continuous on ∂U. □

We have omitted the explicit probabilities and bounds towards the end of the proof, and these bounds have been made explicit in [5] and [8].

We can now combine harmonic functions with the discussion in section 3 to get a result important for proving Liouville's.

Theorem 4.11. *Let $U \subseteq \mathbb{R}^d$ be open and connected and $f : U \to \mathbb{R}$ be harmonic. Suppose $\{B(t) : t \geq 0\}$ is Brownian motion starting inside U and stopping at time T. Then $\{f(B(t)) : t \geq 0\}$ is a local martingale process.*

4.3. **Liouville's Theorem.** The widely-known theorem from all introductory complex analysis courses states that bounded, entire functions must be constant. Using all the machinery we have developed, we are finally able to give a proof later in this section of this result using the results from analyzing Brownian motion.

Recall from Section 2 the concepts of transience and recurrence. We had to put off the proofs because we had not explored the links between Brownian motion and the Dirichlet problem, but now we are ready to prove this in detail. As alluded to in the closing statements of Section 2, the transient and recurrent behavior of Brownian motion depends on the dimension of the space it is in. In $\mathbb{R}^2$, one pictures Brownian motion as a random path so taking all possible paths and allowing them to evolve infinitely, we should fill the space. Naively, one may also extend this into higher dimensions, but we see this is not the case.

When analyzing recurrence of Brownian motion, we often consider its exit probability from annuli of the form $A = \{x : |x| \in (r, R)\} \subseteq \mathbb{R}^d$ for $0 < r < R$. Immediately one sees the connection with the Dirichlet problem.

Define stopping times

$$T_r := \inf\{t > 0 : |B(t)| = r\}$$

to be the first time the Brownian motion B hits the boundary of the r-ball centered at the origin. Then B will exit the annulus A for the first time at $T = \min\{T_r, T_R\}$.

Lemma 4.12. *Continuing with the notation from the previous section, we have*

$$P(T_r < T_R) = \frac{u(R) - u(x)}{u(R) - u(r)}.$$

Proof. The annulus satisfies the Poincaré cone condition, so applying the Dirichlet problem with boundary conditions $u : \overline{A} \to \mathbb{R}$ restricted to ∂A, we get the result. ☐

To turn this lemma into explicit solutions of the boundary condition, we can let u be fixed on the boundaries of the annulus. A simple computation shows that

$$u(x) = \begin{cases} |x|, & d = 1 \\ 2\log|x|, & d = 2 \\ |x|^{2-d}, & d \geq 3 \end{cases}$$

is an explicit solution to the Dirichlet problem with these boundary conditions.

Combining this with the previous lemma, we see that if $\{B(t) : t \geq 0\}$ is a Brownian motion starting at $x \in A$, then plugging u into the expression of P in Lemma 4.12 yields

$$P(T_r < T_R) = \begin{cases} \frac{R-|x|}{R-r}, & d = 1 \\ \frac{\log(R/|x|)}{\log(R/r)}, & d = 2 \\ \frac{R^{2-d}-|x|^{2-d}}{R^{2-d}-r[2-d}, & d \geq 3 \end{cases}.$$

Sending $R \to \infty$, we find for any $|x| > r$,

$$P(T_r < \infty) = \begin{cases} 1, & d < 3 \\ \left(\frac{|x|}{r}\right)^{2-d}, & d \geq 3 \end{cases}.$$

Theorem 4.13 (Transience and Recurrence of Brownian Motion)**.** *Let $B = \{B(t) : t \geq 0\}$ be a Brownian motion in $\mathbb{R}^d$.*

a) If $d = 2$, then B is neighborhood recurrent.
b) If $d \geq 3$, then B is transient.

Proof. First consider the case when $d = 2$. Let $\varepsilon > 0$ and $x \in \mathbb{R}^2$. By Theorem 2.7, we can get a stopping time

$$t_1 = \inf\{t > 0 : B(t) \in B(x, \varepsilon)\}.$$

By the preceeding discussion, $t_1 < \infty$ almost surely. Now consider $t_1 + 1$. By Theorem 4.4, we get another almost surely finite stopping time

$$t_2 = \inf\{t > t_1 + 1 : B(t) \in B(x, \varepsilon)\}.$$

Proceeding inductively, we get an increasing sequence of stopping times such that $B(t_n) \in B(x, \varepsilon)$.

The case when $d \geq 3$ follows from Theorems 2.1 and 4.4. For details, see [5]. □

Putting everything together, we are finally able to prove Liouville's.

Theorem 4.14 (Liouville's)**.** *A bounded, entire function f is constant.*

Proof. Let f be entire. Suppose that f is nonconstant. By Theorem 4.11, $f \circ B$ is a continuous local martingale. By f $\circ B$ is a standard Brownian motion. And by Theorem 4.11, $f \circ B$ is dense in $\mathbb{C}$ and thus f cannot be bounded. □

5. Conclusion and Closing Remarks

This concludes our discussion of Brownian motion, stochastic calculus, and complex analysis. Stochastic calculus is an incredibly rich field of study, and the handful of pages dedicated to its discussion here were nowhere near enough to fully develop its theory and ideas. Nevertheless, we hope that it provided enough background to demonstrate the relationship between Brownian motion, the Dirichlet problem, and Liouville's theorem. In fact, one more result left unproven here is the existence of Green's functions, which is a highly nontrivial yet crucial fact used through some fields of complex analysis. This too can be proven using the theory of Brownian motion, and proofs can be found in [5].

Far outside the scope of our discussion is the application of Brownian motion to other fields of study, such as physics, chemistry, and economics. This was the primary motivation of the development of this idea, and no discussion of Brownian motion would be complete without at least mentioning this. Throughout our discussion here, we have used our intuition with the physical meaning of Brownian motion to guide our understanding of the varying results in each section, and this alone should demonstrate merit in further research in its physical applications.

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