2011 Fall Analysis Qual

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1. Consider the sets

$$\mathcal{E}_n(k) = \bigcap_{m \geq n} \left\{ x : |f_m(x) - f(x)| < \frac{1}{k} \right\}.$$

We have that for any fixed k, $\mathcal{E}_n(k)$ is an increasing sequence (obvious) and

$$\bigcup_{n=1}^{\infty} \mathcal{E}_n(k) \quad \text{is almost all of } [0,1].$$

This is because $x \in \bigcup_{n=1}^{\infty} \mathcal{E}_n(k) \iff x \in \mathcal{E}_n(k)$ for some $n \iff |f_m(x) - f(x)| < \frac{1}{k}$ $\forall m \geq n$ for some $n \iff f_n(x) \to f(x)$.

But $f_n \to f$ a.e., so $\bigcup_{n=1}^{\infty} \mathcal{E}_n(k)$ covers almost everything.

Since $\left| \bigcup_{n=1}^{\infty} \mathcal{E}_n(k) \right| = 1$, we have that for any k, given a fixed $\varepsilon > 0$, $\exists n$ s.t. $|\mathcal{E}_n(k)^c| < \varepsilon$.

Now fix $\varepsilon > 0$. $\forall j \geq 1$, $\exists n_j$ such that

$$|\mathcal{E}_{n_j}(j)^c| < \varepsilon \cdot 2^{-j} \quad \text{by above.}$$

Consider $\mathcal{E} = \bigcap_{j=1}^{\infty} \mathcal{E}_{n_j}(j)$; then

$$|\mathcal{E}^c| = \left| \bigcup_{j=1}^{\infty} \mathcal{E}_{n_j}(j)^c \right| \leq \sum_{j=1}^{\infty} |\mathcal{E}_{n_j}(j)^c| < \sum_{j=1}^{\infty} \varepsilon 2^{-j} = \varepsilon.$$

Moreover, $f_n \to f$ uniformly on $\mathcal{E}$: $\forall j \geq 1$, $\exists n_j > 0$ (as constructed above) s.t. $\mathcal{E}_{n_j}(j) \supseteq \mathcal{E}$; thus $\forall n \geq n_j$,

$$|f_n(x) - f(x)| < \frac{1}{j} \quad \forall x \in \mathcal{E} \subseteq \mathcal{E}_{n_j}(j). \qquad \square$$

6. (X, d) is a compact metric space, so its T^4.

Fix $\varepsilon > 0$.

Let $K \subseteq X$ compact. Since μ is a finite Borel measure, $\exists K' \subset K^c$ compact s.t. $\mu(X \setminus K') < \mu(K) + \varepsilon$.

$\exists f \in C(X)$ s.t. $f = 1$ on K & $f = 0$ on K'.

$$\int_X f \, d\mu < \int_{X \setminus K'} d\mu = \mu(K) + \varepsilon.$$

Since $\int_X f \, d\mu_n \geq \int_K d\mu_n = \mu_n(K)$ $\forall n$, we get

$$\limsup_{n \to \infty} \mu_n(K) \leq \lim_{n \to \infty} \int_X f \, d\mu_n$$

$$= \int_X f \, d\mu < \mu(K) + \varepsilon.$$

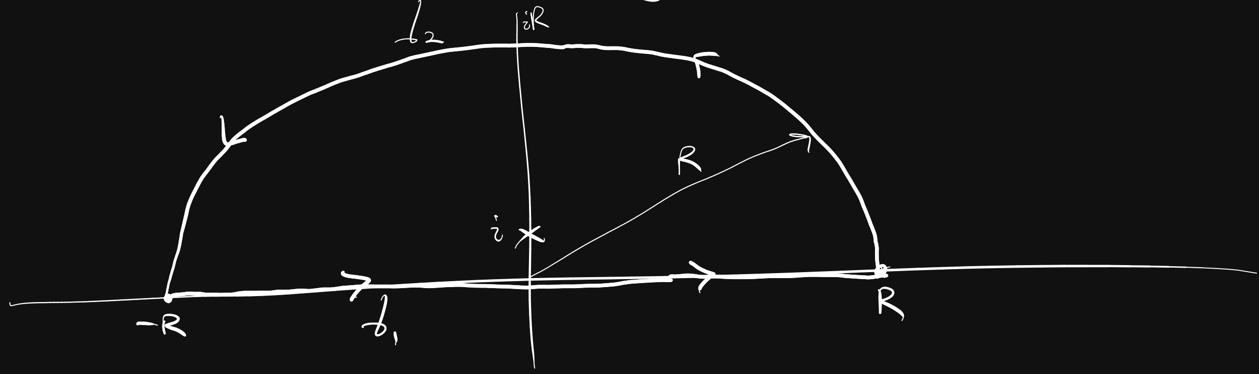
ε was arbitrary, so we conclude

$$\mu(K) \geq \limsup_{n \to \infty} \mu_n(K). \quad \square$$

7. Consider that $\cos x = \mathrm{Re}(e^{ix})$; hence

$$\int_0^\infty \frac{\cos x}{(1+x^2)^2}\,dx = \mathrm{Re}\left(\int_0^\infty \frac{e^{ix}}{(1+x^2)^2}\,dx \right).$$

Define $f(z) = \dfrac{e^{iz}}{(1+z^2)^2}$, which has double poles at $z = \pm i$. Consider the following semicircular contour:



Let γ_1 be the line segment from $-R$ to R, and let γ_2 be the semicircular arc pictured above. Let γ be the concatenation of the two. Then, by the residue theorem,

$$\frac{1}{2\pi i} \oint_\gamma f(z)\,dz = \mathrm{Res}_f(i)$$

$$= \lim_{z \to i} (z-i)^2 \cdot \frac{e^{iz}}{(z^2+1)^2}$$

$$= \lim_{z \to i} \frac{e^{iz}}{(z+i)^2}$$

$$= \frac{e^{-1}}{(i-1)^2} = \frac{1}{-2ie}$$

Taking limits,

$$\lim_{R \to \infty} \oint_\gamma f(z)\,dz = -\frac{\pi}{e}.$$

As $R \to \infty$, $\int_{\gamma_1} f(z)\,dz \to \int_{-\infty}^{\infty} f(z)\,dz$. On the other hand, when z is on γ_2, $|z| = R$ and $|e^{iz}| \le 1$ — γ_2 is in the upper half-plane. Hence,

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_{\gamma_2} \frac{|e^{iz}|}{(1+|z|^2)^2}\,dz \le \int_{\gamma_2} \frac{1}{(1+R^2)^2}\,dz$$

$$= \frac{\pi R}{(1+R^2)^2} \longrightarrow 0.$$

Thus,

$$\int_{-\infty}^{\infty} f(z)\,dz = -\frac{\pi}{e}$$

$$\operatorname{Re}\int_{-\infty}^{\infty} f(z)\,dz = \int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\,dx \xrightarrow{\text{integrand is even}} 2\int_{0}^{\infty} \frac{\cos x}{(1+x^2)^2} = -\frac{\pi}{e}$$

$$\boxed{\int_{0}^{\infty} \frac{\cos x}{(1+x^2)^2}\,dx = \frac{\pi}{2e}}$$

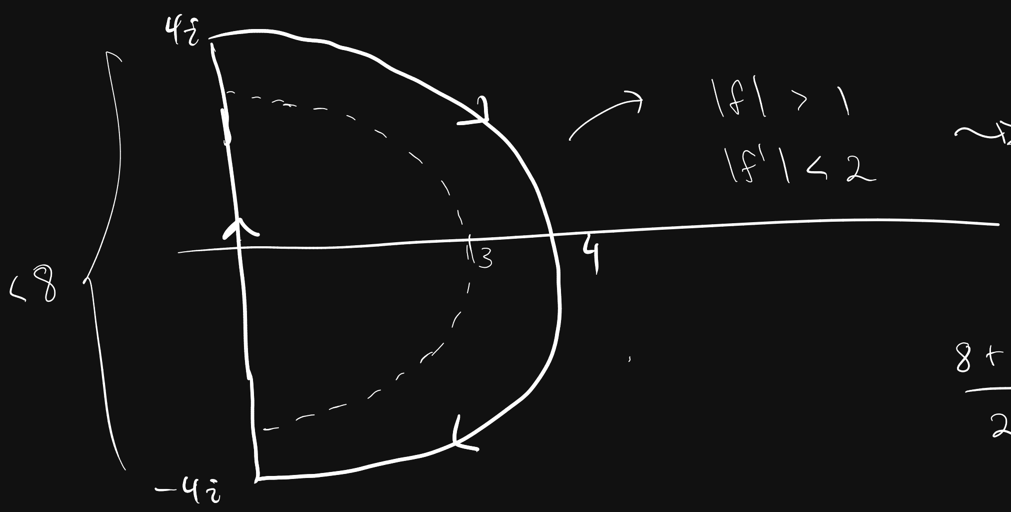
8. Let $f(z) = z - 2 - e^{-z}$; $f'(z) = 1 + e^{z}$.

For $x \geq 0$ real, we have $f'(x) = 1 + e^{x} > 0$; hence by Rolle's theorem there's at most one real zero. Additionally, $f(0) = -3$ while $\lim_{x \to \infty} f(x) = +\infty$; hence by the intermediate value theorem $f(x)$ has at least one ~~positive~~ real zero. Thus, f has exactly one real zero, which is positive.

Next, note $\overline{f(z)} = \overline{z - 2 + e^{-z}} = \bar{z} - 2 + e^{-\bar{z}} = f(\bar{z})$; thus if $f(z) = 0$, $f(\bar{z}) = 0$. Nonreal solutions come in pairs, so f has an odd # of zeros.

Now if $f(z) = 0$, $z = 2 + e^{-z} \Rightarrow |z| \leq |2| + |e^{-z}| < 3$. Note $|e^{-z}| = e^{-\operatorname{Re} z} < 1$ if $z \in H$. Moreover, if z is purely imaginary, then $\operatorname{Re}(z - 2 - e^{-z}) = -2 - e^{-\operatorname{Re} z} \neq 0$, so no zeroes of f lie on the imaginary axis.

Consider this contour:



$|f| > 1$

$|f'| < 2$

$\rightsquigarrow < \dfrac{8\pi}{R-3}$ $\qquad \dfrac{R}{\pi} + \dfrac{R}{R-3}$

$\dfrac{8 + 8\pi}{2\pi} = \dfrac{4}{\pi} + 4$

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f}\, dz = \# \text{ of zeroes inside } \gamma = \# \text{ of zeroes of } f \text{ in } H,$$

as f has no poles & no zeroes lie on γ.

We have $f' = 1 + e^{-z}$, so

9. $$\int_{D^*} |f(z)|^2 \, d\lambda(z)$$

$$w = \frac{1}{z}, \qquad d\lambda(w) = \left|\frac{1}{z^2}\right|^2 d\lambda(z) \qquad d\lambda(z) = \left|\frac{1}{w}\right|^2 d\lambda(w)$$

$$= \int_{|w|>1} \left|\frac{1}{w} f\left(\frac{1}{w}\right)\right|^2 d\lambda(w) \qquad < \infty$$

Thus $\displaystyle\lim_{|w|\to\infty} \left|\frac{1}{w} f\left(\frac{1}{w}\right)\right| = \lim_{|z|\to 0} |z f(z)| = 0,$

and it follows that f has a removable singularity at the origin. $\square$

10. By the open mapping theorem, $\exists \phi : \Omega \xrightarrow{\sim} \mathbb{D}$ biholomorphic/conformal.

Consider

$$\mathbb{D} \xrightarrow{\phi^{-1}} \Omega \xrightarrow{f} \Omega \xrightarrow{\phi} \mathbb{D}$$

$$\tilde{f} = \phi \circ f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D};$$

f is identity $\iff$ $\tilde{f}$ is identity, and $\tilde{f}$ also fixes two distinct points. By composing with an additional automorphism of $\mathbb{D}$, we may assume wolog $\Omega = \mathbb{D}$, $z_2 = 0$.

However, we know that if $f : \mathbb{D} \to \mathbb{D}$ s.t. $f(0) = 0$, then $|f(z)| \leq |z|$ $\forall z \in \mathbb{D}$.