ANALYSIS QUAL: SEPTEMBER 15, 2022

Please be reminded that to pass the exam you need to show mastery of both real and complex analysis. Please choose at most 10 questions to answer, including at least 4 from problems 1–6 and 4 from problems 7–12. On the front of your paper indicate which 10 problems you wish to have graded.

Problem 1. Let $f \in L^1(\mathbb{R}^d)$ and let $a \in \mathbb{R}^d \setminus \{0\}$. For each $k \in \mathbb{N}$, determine the limit

$$\lim_{t\to\infty} \int \left| \sum_{j=1}^{k} f(jx + ta) \right| dx.$$

Problem 2. Let $f \in L^p(\mathbb{R})$, for some $1 \leq p < 2$. Show that the series

$$\sum_{n=1}^{\infty} \frac{f(x+n)}{\sqrt{n}}$$

converges absolutely for almost all $x \in \mathbb{R}$. For each $2 \leq p \leq \infty$, give an example of a function $f \in L^p(\mathbb{R})$ for which the series diverges for every $x \in \mathbb{R}$.

Problem 3. Fix $K > 0$ and let $\mathcal{M}_K(\mathbb{R}^d)$ denote the space of finite positive Borel measures μ on $\mathbb{R}^d$ with $\mu(\mathbb{R}^d) \leq K$. When $\mu_1, \mu_2 \in \mathcal{M}_K(\mathbb{R}^d)$, write $\mu_1 \leq \mu_2$ if $\mu_1(U) \leq \mu_2(U)$ for each Borel set $U \subset \mathbb{R}^d$. Show that the set

$$\{(\mu_1, \mu_2) \in \mathcal{M}_K(\mathbb{R}^d) \times \mathcal{M}_K(\mathbb{R}^d) : \mu_1 \leq \mu_2\}$$

is compact for the weak* topology. Here $\mathcal{M}_K(\mathbb{R}^d)$ is viewed as a subset of the space of finite Borel measures on $\mathbb{R}^d$, equipped with the weak* topology.

Problem 4. Suppose $2 \leq p < \infty$. If μ, ν are positive measures on $\mathbb{R}^d$ and f, f_j are finitely many functions in $L^p(\mathbb{R}^d)$ satisfying $f = \sum_j f_j$, the inequality

$$\|f\|_{L^p(\mu)} \leq M \left(\sum_j \|f_j\|^2_{L^p(\nu)} \right)^{1/2} \tag{1}$$

is called an $\ell^2 L^p$ decoupling inequality and $M > 1$ is called the decoupling constant. Show that if $\mu = \sum_k \mu_k$ and $\nu = \sum_k \nu_k$, where the sums are finite, and the $\ell^2 L^p$ decoupling inequalities hold with decoupling constant M,

$$\|f\|_{L^p(\mu_k)} \leq M \left(\sum_j \|f_j\|^2_{L^p(\nu_k)} \right)^{1/2},$$

for all k, then (1) holds with the same decoupling constant M.

Problem 5. Let us define $Tf(x) = \int_0^x f(t)dt$, $x \in [0,1]$, for $f \in L^2([0,1])$.

(a) Prove that $T : L^2([0,1]) \to L^2([0,1])$ is a linear continuous map which is *compact*, in the sense that for any bounded sequence $f_n \in L^2([0,1])$, the sequence Tf_n has a convergent subsequence in $L^2([0,1])$.

(b) Prove that T has no eigenvalues, i.e. prove that there is no $\lambda \in \mathbb{C}$ such that $Tf = \lambda f$ for some nonzero $f \in L^2([0,1])$.

(c) Show that the spectrum of T is $\{0\}$, i.e. show that the map $f \mapsto Tf - \lambda f$ is an isomorphism of $L^2([0,1])$ for each $0 \neq \lambda \in \mathbb{C}$, and that it is not an isomorphism of $L^2([0,1])$ for $\lambda = 0$.

Problem 6. Let $E = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 \notin \mathbb{Q}\}$. Show that E does not contain a set of the form $A_1 \times A_2$, where $A_1 \subset \mathbb{R}$, $A_2 \subset \mathbb{R}$ are measurable, both of positive Lebesgue measure.

Problem 7. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

(a) Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic injective and let $g : \mathbb{D} \to \mathbb{C}$ be holomorphic such that $g(0) = f(0)$ and $g(\mathbb{D}) \subseteq f(\mathbb{D})$. Show that $g(\mathbb{D}_r) \subseteq f(\mathbb{D}_r)$, for each $0 < r < 1$. Here $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$.

(b) Let $g : \mathbb{D} \to \mathbb{C}$ be holomorphic such that $g(0) = 0$ and $|\operatorname{Re} g(z)| < 1$ for all $z \in \mathbb{D}$. Show that

$$|\operatorname{Im} g(z)| \leq \frac{2}{\pi} \log \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D}.$$

Problem 8. Show that

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2},$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$, with the series in the right hand side converging uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

Problem 9. Let $\Omega \subset \mathbb{C}$ be open connected and let $f_j : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions. Suppose that $f_j(a)$ converges as $j \to \infty$, for some $a \in \Omega$, and that the sequence $\operatorname{Re} f_j$ converges as $j \to \infty$, uniformly on compact subsets of Ω. Show that f_j converges as $j \to \infty$, uniformly on compact subsets of Ω.

Problem 10. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and set

$$m(r) = \frac{1}{2\pi} \int_0^{2\pi} \log_+ |f(re^{i\varphi})| \, d\varphi.$$

Here $\log_+ t = \max(\log t, 0)$. Suppose that

$$\limsup_{r \to \infty} \frac{m(r)}{\log r} < \infty.$$

Show that f is a polynomial.

Problem 11. Let $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ and define

$$u(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z}\, dt, \quad \operatorname{Im} z \neq 0.$$

(a) Prove that u is a holomorphic function on $\mathbb{C} \backslash \mathbb{R}$ such that $u(z) \to 0$ as $|\operatorname{Im} z| \to \infty$.

(b) Show that the limit

$$\lim_{y \to 0^+} \left(u(z) - u(\overline{z}) \right), \quad z = x + iy,$$

exists for each $x \in \mathbb{R}$ and compute it.

Problem 12. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and assume that f is *not* of the form $z \mapsto z + a$ for some $a \in \mathbb{C}$.

(a) Show that the composition $f \circ f$ has a fixed point.

(b) Find an entire $f : \mathbb{C} \to \mathbb{C}$ (not of the form $z \mapsto z + a$) with no fixed points.

Hint: One approach centers on the function $z \mapsto [f(f(z)) - z]/[f(z) - z]$.