Analysis Qual, F22

1. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$, for any $\varepsilon > 0$, $\exists g \in C_c^\infty(\mathbb{R}^d)$ s.t. $\|g-f\|_1 < \varepsilon$. We claim 2 things:

1. $$\lim_{t\to\infty} \int \left|\sum_{j=1}^k f(jx+at)\right| dx \approx \lim_{t\to\infty} \int \left|\sum_{j=1}^k g(jx+at)\right| dx$$

2. $$\lim_{t\to\infty} \int \left|\sum_{j=1}^k g(jx+at)\right| dx = \sum_{j=1}^k \frac{1}{j} \|g\|_1$$

For the first claim, we see that for any t, we have

$$\int \left|\sum_{j=1}^k f(jx+at)\right| - \left|\sum_{j=1}^k g(jx+at)\right| dx$$

$$\leq \int \left|\sum_{j=1}^k f(jx+at) - g(jx+at)\right| dx$$

$$\leq \sum_{j=1}^k \int |f(jx+at) - g(jx+at)| dx$$

$$= \sum_{j=1}^k \frac{1}{j} \|f-g\|_1 \qquad < \varepsilon \sum_{j=1}^k \frac{1}{j}.$$

For the second claim, $\exists R > 0$ s.t. $\operatorname{supp}(g) \subseteq B(0,R)$. Then, if $t > \frac{cR}{\|a\|_{\mathbb{R}^d}}$, then $\|ta\|_{\mathbb{R}^d} > cR$, and $jx + at \in \operatorname{supp} g$ implies $\|jx\| > cR$. Thus, $\|x\| > \frac{c}{j} R$. Picking $c > 2k$ ensures $\|x\| > 2R$ if any of the $jx+at \in \operatorname{supp} g$, so $\forall x \in \mathbb{R}^d$ & t very large, the sum only has one nonvanishing term. Hence,

$$\lim_{t\to\infty} \int \left|\sum_{j=1}^k g(jx+at)\right| dx = \lim_{t\to\infty} \sum_{j=1}^k \int |g(jx+at)| dx = \sum_{j=1}^k \frac{1}{j} \cdot \|g\|_1.$$

Finally, since $\big|\|g\|_1 - \|f\|_1\big| < \varepsilon$, we see that

$$\left| \lim_{t\to\infty} \int \left|\sum_{j=1}^k f(jx+at)\right| dx - \sum_{j=1}^k \frac{1}{j} \|f\|_1 \right|$$

$$\leq \left| \lim_{t\to\infty} \int \left|\sum_{j=1}^k f(jx+at)\right| dx - \int \left|\sum_{j=1}^k g(jx+at)\right| dx \right| + \left| \lim_{t\to\infty} \int \left|\sum_{j=1}^k g(jx+at)\right| dx - \sum_{j=1}^k \frac{1}{j}\|g\|_1 \right| + \left| \sum_{j=1}^k \frac{1}{j} (\|g\|_1 - \|f\|_1) \right|$$

$$= 2\varepsilon \sum_{j=1}^k \frac{1}{j}. \quad \text{Send } \varepsilon \to 0. \quad \square$$

2. Let q conjugate to p; note $2 < q \leq \infty$.

We claim that $\forall N \in \mathbb{Z}$, the series converges for a.e. $x \in [N, N+1)$. This implies it converges abs'ly a.e. on $\mathbb{R}$. For this, consider that

$$\int_N^{N+1} \sum_{n=1}^{\infty} \left| \frac{f(x+n)}{\sqrt{n}} \right| dx \overset{MCT}{=} \sum_{n=1}^{\infty} \int_N^{N+1} \left| \frac{f(x+n)}{\sqrt{n}} \right| dx$$

Now write $g_N = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \chi_{[N+n,\, N+n+1]}$. Then, we get

$$= \sum_{n=1}^{\infty} \int_N^{N+1} \left| f(x+n)\, g(x+n) \right| dx$$

$$= \int_N^{\infty} \left| f(x+n)\, g(x+n) \right| dx$$

$$= \int_{\mathbb{R}} |fg|\, dx.$$

But $f \in L^p$ & $g \in L^q$:

$$\|g\|_q^q = \sum_{n=1}^{\infty} n^{-q/2} < \infty \quad \text{since } \frac{q}{2} > 1.$$

Thus, by Hölder,

$$\int_N^{N+1} \sum_{n=1}^{\infty} \left| \frac{f(x+n)}{\sqrt{n}} \right| dx < \infty \Rightarrow \text{integrand is finite for a.e. } x \in [N, N+1].$$

$\square$

3. The space of finite Borel measures is the dual of $L^\infty(\mathbb{R}^d)$. By Alaoglu's theorem, $M_K(\mathbb{R}^d)$ (the closed ball of rad. K in $L^\infty(\mathbb{R}^d)^*$) is weak* - compact, so $M_K(\mathbb{R}^d) \times M_K(\mathbb{R}^d)$ is also weak* - compact. It thus suffices to show that the set M is weak*-closed.

Suppose $\{(\mu_n, \nu_n)\}$ is a sequence of finite Borel measures, each with $\int d\mu_n, \int d\nu_n \leq K$ and $\mu_n \leq \nu_n$, converging to some (μ, ν) in weak*. $\forall$ Borel $\mathcal{U}$, we have

$$\int_{\mathcal{U}} d\mu_n \longrightarrow \int_{\mathcal{U}} d\mu \quad \text{and} \quad \int_{\mathcal{U}} d\nu_n \longrightarrow \int_{\mathcal{U}} d\nu.$$

But $\int_{\mathcal{U}} d\mu_n \leq \int_{\mathcal{U}} d\nu_n \; \forall n$, so $\int_{\mathcal{U}} d\mu \leq \int_{\mathcal{U}} d\nu$.

Thus $\mu \leq \nu$ and the set is closed. Closed subsets of compact spaces are compact. $\square$

4. We have

$$\|f\|^p_{L^p(\mu)} = \sum_k \|f\|^p_{L^p(\mu_k)} \le M^p \sum_k \left(\sum_j \|f_j\|^2_{L^p(\nu_k)} \right)^{p/2}.$$

Consider the function $g(j,k) = \|f_j\|^2_{L^p(\nu_k)}$ $\forall j,k$. Viewing the sums as integrals w.r.t. a counting measure, and using $1 \le p/2 < \infty$, Minkowski's integral inequalities give

$$
\begin{aligned}
\|f\|^{2/p}_{L^p(\mu)} &\le M^{2/p} \sqrt{\left[\sum_k \left(\sum_j g(j,k) \right)^{p/2} \right]^{2/p}} \\
&\le M^{2/p} \sqrt{\sum_j \left(\sum_k g(j,k)^{p/2} \right)^{2/p}} \\
&= M^{2/p} \sqrt{\sum_j \left(\sum_k \|f_j\|^p_{L^p(\nu_k)} \right)^{2/p}} \\
&= M^{2/p} \left(\sum_j \|f_j\|^2_{L^p(\nu)} \right)^{1/p}.
\end{aligned}
$$

Rewrite exponents. $\square$

5. a) Linearity of T is immediate from linearity of integrals. It suffices to show T is bounded. Let $f \in L^2([0,1])$ s.t. $\|f\|_2 = 1$. Then,

$$\|Tf(x)\|_2 = \left\| \int_0^1 \chi_{[0,x]} f \, dt \right\|_2$$

$$= \left(\int_0^1 \left| \int_0^1 \chi_{[0,x]} f \, dt \right|^2 dx \right)^{1/2}$$

Minkowski's integral inequality

$$\leq \int_0^1 \left(\int_0^1 \chi_{[t,1]} f^2 \, dx \right)^{1/2} dt$$

$$\leq 1.$$

Thus T is bounded & hence cts. If f_n is a bounded sequence in $L^2([0,1])$, say w/ $\|f_n\|_2 \leq M \; \forall n$, then Tf_n is a sequence in $B_{L^2([0,1])}(0, M)$. Since L^2 is self-dual, Alaoglu gives our weakly convergent subseq.

6. We claim that if $A_1, A_2 \subseteq \mathbb{R}$ are (Lebesgue) measurable & non-negligible, then

$$A_1 - A_2 := \{ x_1 - x_2 : x_1 \in A_1, x_2 - A_2 \} \subseteq \mathbb{R}$$

contains an open interval. In particular, it contains a <u>rational</u>, so $\nexists$ such A_1 & A_2 s.t. $A_1 \times A_2 \subseteq \mathcal{E}$.

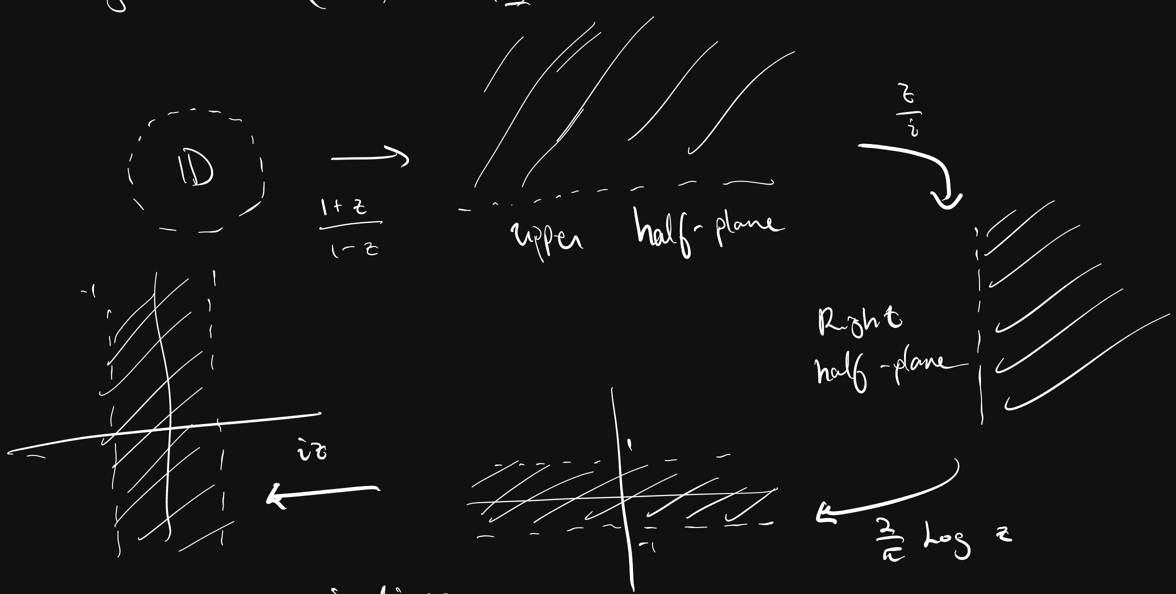
7. a) $f^{-1} \circ f$ is identity on disc.

Since $\operatorname{im} g \subseteq \operatorname{im} f$, $f^{-1} \circ g$ is a map of the disc to itself. By Schwarz lemma, $|f^{-1} \circ g(z)| \leq |z|$ $\forall z \in \mathbb{D}$. In particular, $|z| < r$ implies

$$|f^{-1} \circ g(z)| < r \Rightarrow f^{-1} \circ g(z) \in f^{-1} \circ f(\mathbb{D}_r).$$

By injectivity of f, it follows $g(z) \in f(\mathbb{D}_r)$ $\forall z \in \mathbb{D}_r$, i.e. $g(\mathbb{D}_r) \subseteq f(\mathbb{D}_r)$.

b)



Putting these injective maps together, we get

$$\varphi : \mathbb{D} \longrightarrow \{ z \in \mathbb{C} : |\operatorname{Re} z| < 1 \}$$

$$z \longmapsto \frac{2i}{\pi} \operatorname{Log}\left(\frac{1}{i} \cdot \frac{1+z}{1-z} \right).$$

(We pick a good branch of Log)

Clearly, $g(\mathbb{D}) \subseteq \varphi(\mathbb{D})$ by assumption, and $\varphi(0) = 0 = g(0)$.

Let $z \in \mathbb{D}$, and let $r > 0$ s.t. $|z| < r < 1$, i.e. $z \in \mathbb{D}_r$. Then,

$$\left| \operatorname{Im}(\varphi(z)) \right| = \frac{2}{\pi} \left| \operatorname{Re}\left(\operatorname{Log}\left(\frac{1}{i} \cdot \frac{1+z}{1-z} \right) \right) \right|$$

$$= \frac{2}{\pi} \log\left(\frac{1+|z|}{1-|z|} \right)$$

$$< \frac{2}{\pi} \log\left(\frac{1+r}{1-r} \right).$$

In particular, since $g(\mathbb{D}_r) \subseteq \varphi(\mathbb{D}_r)$, we have

$$\left|\operatorname{Im}(g(z))\right| \leq \left|\operatorname{Im}(\varphi(z))\right|$$

$$< \frac{2}{\pi} \log\left(\frac{1+r}{1-r}\right) \qquad \forall\, |z| < r < 1.$$

Let $r \to |z|^+$ to get the claim. $\square$

9. Let $z \in \Omega$. Ω open & connected $\Rightarrow$ Ω path connected, so $\exists\, \gamma$ a path from a to z. Let K a compact nbhd of image of γ. $\operatorname{Re} f_j$ converges unif'ly on K, so in particular

$\partial f_j / \partial x$ converges uniformly

11. a) Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$t - z = (t - \mathrm{Re}\, z) + i\, \mathrm{Im}\, z \Rightarrow |t - z| \geq |\mathrm{Im}\, z| \quad \forall t \in \mathbb{R}.$$

Thus, the integral

$$\int_{-\infty}^{\infty} \left| \frac{f(t)}{t - z} \right| dt \leq \frac{1}{|\mathrm{Im}\, z|} \int_{-\infty}^{\infty} |f(t)| dt = \frac{\|f\|_{L_1}}{|\mathrm{Im}\, z|} < \infty$$

converges absolutely, so u is holo. at z $\forall z \in \mathbb{C} \setminus \mathbb{R}$.

By the same bound, $u \to 0$ as $|\mathrm{Im}\, z| \to \infty$.

b)

$$\frac{1}{t + iy} - \frac{1}{t - iy} = \frac{t - iy - t - iy}{t^2 + y^2} = -\frac{2iy}{t^2 + y^2}.$$

Thus, when expanding

$$u(z) - u(\bar{z}) = \frac{1}{2\pi i} \int_{\mathbb{R}} f(t - x) \frac{-2iy}{t^2 + y^2} dt$$

$$= -\frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t - x)}{t^2 + y^2} dt$$

$f \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \Rightarrow |f(t)| \leq M$ for some $M > 0$.

Thus we get

$$|u(z) - u(\bar{z})| \leq \frac{y}{\pi} M \int_{\mathbb{R}} \frac{1}{t^2 + y^2} dt$$

$$= \frac{y}{\pi} \cdot M \cdot \frac{\pi}{y} = M.$$

12. a) Suppose $f \circ f$ has <u>no</u> f.p.'s. Then f has no f.p.'s either; consider the function

$$g(z) = \frac{f(f(z)) - z}{f(z) - z},$$

which is entire & avoids zero.