Math 131BH Final Practise Problems

1. Define the sequence of functions $\{f_n(x)\}_{n=0}^{\infty}$ on $[1,\infty)$ by $f_0(x) = x$ and

$$f_{n+1}(x) = \frac{1}{2}\left(f_n(x) + \frac{x}{f_n(x)}\right).$$

Show that for any compact set $K \subseteq [1,\infty)$, $\{f_n(x)\}$ converges uniformly on K. Does $\{f_n\}$ converge uniformly on $[1,\infty)$?

2. (a) Let $f : \mathbb{R} \to \mathbb{R}$ continuous, and suppose that for every odd function $\phi \in C_0^\infty(\mathbb{R})$ (i.e., $\phi(x) = -\phi(-x)$ for all x) one has

$$\int f(x)\phi(x)dx = 0.$$

Prove that $f(x) = f(-x)$ for all x.

(b) Let $f \in C_0^\infty(\mathbb{R})$, and suppose one has

$$\int f(x)\exp\left(ax^2\right)dx = 0$$

for all $a \in \mathbb{R}$. Show that f is odd, i.e. $f(x) = -f(-x)$ for all x.

Hint: Show that

$$\int_{-R}^{R} f(x)\exp\left(ax^2\right)dx = \int_0^R \left(f(x) + f(-x)\right)\exp\left(ax^2\right)dx$$

for any $R > 0$ and $a \in \mathbb{R}$, then argue that this implies $f(x) + f(-x) = 0$ on $[0,\infty)$.

3. Let $\{f_n\}$ be a sequence of bounded continuous functions $\mathbb{R} \to \mathbb{R}$. Let $M_n = \sup_{x\in\mathbb{R}} |f_n(x)|$ and suppose $\sum_{n=1}^{\infty} M_n$ converges. Prove that $F(x) = \prod_{n=1}^{\infty}(1 + f_n(x))$ converges for each x and is continuous. Here, the infinite product is defined as

$$\prod_{n=1}^{\infty}(1 + f_n(x)) := \lim_{N\to\infty}\prod_{n=1}^{N}(1 + f_n(x)).$$

4. Let $0 < \alpha \le 1$. Recall the space of α-Hölder continuous functions on $[0,1]$, denoted $C^\alpha([0,1])$, the set of continuous functions $f : [0,1] \to \mathbb{R}$ satisfying

$$\|f\|_{C^\alpha} := \sup_{x\in[0,1]} |f(x)| + \sup_{x\neq y}\frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

Let $S = \{f \in C^\alpha([0,1]) : f(0) = 0\}$. Show that S is a closed subset of $C^\alpha([0,1])$. Is S compact?

5. Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, x^2 + y^2 \leq 1\}$.

   (a) Prove that for any $\epsilon > 0$ and any continuous function $f : \mathbb{H} \to \mathbb{R}$ there exists a function $g(x, y)$ of the form

   $$g(x, y) = \sum_{m=0}^{N} \sum_{n=0}^{N} a_{mn} x^{2m} y^{2n}$$

   for some natural number N and real numbers a_{mn} satisfying

   $$\sup_{(x,y)\in\mathbb{H}} |f(x, y) - g(x, y)| < \epsilon.$$

   (b) Does the result still hold if $\mathbb{H}$ is replaced by the disc $\mathbb{D} = \{(x, y) : x^2 + y^2 \leq 1\}$?

6. (UCLA Basic Exam, Fall 2021)

   Let a_n be any sequence of real numbers. Show that the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ defined by

   $$f_n(x) = \int_0^x \exp\left(t^8 - 6\cos^2(a_n t)\right) dt$$

   has a subsequence that converges uniformly on $[0, 1]$.

7. (UCLA Basic Exam, Fall 2020)

   Let S be a subset of $\mathbb{R}^n$. Prove that the following are equivalent:

   (i) For each $p \in S$, there exists an open subset $V \subseteq \mathbb{R}^n$ containing p, an open subset $U \subseteq \mathbb{R}^{n-1}$, and an injective C^1 function $r : U \to \mathbb{R}^n$ such that $r(U) = V \cap S$ and Dr is injective on U.

   (ii) For each $p \in S$, there exists an open neighbourhood $V \subseteq \mathbb{R}^n$ of p and a C^1 function $f : V \to \mathbb{R}$ such that Df is nonzero on $V \cap S$ and $V \cap S = f^{-1}(\{0\})$.